

Lie algebras :

Classification, Deformations and Rigidity

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In the first section we recall some basic notions on Lie algebras. In a second time we study the algebraic variety of complex n -dimensional Lie algebras. We present different notions of deformations : Gerstenhaber deformations, perturbations, valued deformations and we use these tools to study some properties of this variety. Finally we introduce the concept of rigidity and we present some results on the class of rigid Lie algebras.

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1 Lie algebras

1.1 Definition and examples

In this lecture, the considered Lie algebras are complex (sometimes real but this case will be precised).

Definition 1 A Lie algebra \mathfrak{g} is a pair (V, μ) where V is a complex vector space and μ a bilinear map

$$\mu : V \times V \rightarrow V$$

satisfying :

$$\begin{aligned} \mu(X, Y) &= -\mu(Y, X), \quad \forall X, Y \in V, \\ \mu(X, \mu(Y, Z)) + \mu(Y, \mu(Z, X)) + \mu(Z, \mu(X, Y)) &= 0, \quad \forall X, Y, Z \in V. \end{aligned}$$

This last identity is called the Jacobi relation.

1.2 Examples

1. The simplest case is given taking $\mu = 0$. Such a Lie algebra is called abelian.
2. Suppose that V is 2-dimensional. Let $\{e_1, e_2\}$ be a basis of V . If we put

$$\mu(e_1, e_2) = ae_1 + be_2$$

with the condition of skew-symmetry, this map satisfies the Jacobi condition and it is a multiplication of Lie algebra.

3. Let $sl(2, \mathbb{C})$ be the vector space of matrices A of order 2 such that $tr(A) = 0$ where tr indicates the trace of the matrix A . The product

$$\mu(A, B) = AB - BA$$

is well defined on $sl(2, \mathbb{C})$ because $tr(AB - BA) = 0$ as soon as $A, B \in sl(2, \mathbb{C})$. We can see that this product satisfies the Jacobi condition and we have a Lie algebra of dimension 3.

1.3 The Lie algebra of a Lie group

Let G be a (complex) Lie group of dimension n . For every $g \in G$, we denote by L_g the automorphism of G given by

$$L_g(x) = gx.$$

It is called the left translation by g . Its differential map $(L_g)_x^*$ is an isomorphism

$$(L_g)_x^* : T_x(G) \longrightarrow T_{gx}(G)$$

where $T_x(G)$ designates the tangent space at x to G . A vector field X on G is called left invariant if it satisfies

$$(L_g)_x^*(X(x)) = X(gx)$$

for every x and g in G . We can prove that, if $[X, Y]$ denotes the classical bracket of vector fields and if X and Y are two left invariant vector fields of G then $[X, Y]$ is also a left invariant vector field. This shows that the vector space of left invariant vector fields of G is provided with a Lie algebra structure denoted $L(G)$. As a left invariant vector field X is defined as soon as we know its value $X(e)$ to the unity element e of G the vector space $L(G)$ can be identified to $T_e(G)$. For example the Lie algebra of the algebraic group $SL(2, \mathbb{C})$ is isomorphic to $sl(2, \mathbb{C})$. (We shall define the notion of isomorphism later).

From this construction, to each Lie group we have only one associated Lie algebra. But the converse is not true. In fact two Lie groups which are locally isomorphic have the same associated Lie algebra. On a other hand, we can construct from a finite dimensional complex Lie algebra a Lie group. This is a little bit complicated. But from the bracket of a finite dimensional Lie algebra \mathfrak{g} , we can define a local group whose product is given by the Campbell-Hausdorff formula:

$$X.Y = X + Y + 1/2[X, Y] + 1/12[[X, Y], Y] - 1/12[[X, Y], X] + \dots$$

which is a infinite sequence of terms given from the bracket. This local structure can be extended to a global structure of Lie group. Then we have a one-one correspondance between finite dimensional Lie algebra (on \mathbb{C} for example or on \mathbb{R}) and the simply connected, connected Lie groups. We can note that the dimension of the Lie group as differential manifold is equal to the dimension of its Lie algebra as vector space.

1.4 Relation between Lie group and its Lie algebra

In other talks of this school, the notion of linear algebraic groups is often used. If the Lie group is a linear group, that is a Lie subgroup of the Lie group $GL(n, \mathbb{C})$, then its Lie algebra is a Lie subalgebra of the Lie algebra $gl(n, \mathbb{C})$ whose elements are the complex matrices of order n . In this case we can easily construct a map between the Lie algebra and a corresponding Lie group. It is based on the classical exponential map.

The exponential map from the Lie algebra $gl(n, \mathbb{C})$ of the Lie group $GL(n, \mathbb{C})$ to $GL(n, \mathbb{C})$ is defined by the usual power series:

$$Exp(A) = Id + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

for matrices A . If G is any Lie subgroup of $GL(n, \mathbb{C})$ then the exponential map takes the Lie algebra \mathfrak{g} of G into G , so we have an exponential map for all matrix groups.

The definition above is easy to use, but it is not defined for Lie groups that are not matrix groups. The Ado theorem precises that every finite dimensional Lie algebra can be represented as a linear algebra. This means that , for a given n dimensional Lie algebra \mathfrak{g} , there exists an integer N such that the elements of \mathfrak{g} are written as matrices of order N . The problem comes because we do not know N . Moreover , if we write \mathfrak{g} as a subalgebra of $gl(N, \mathbb{C})$, we can exprim the exponential map, but this map depend of the choosen representation of \mathfrak{g} as linear algebra. We can solve both problems using a more abstract definition of the exponential map that works for all Lie groups, as follows. Every vector X in \mathfrak{g} determines a linear map from \mathbb{R} to \mathfrak{g} taking 1 to X , which can be thought of as a Lie algebra homomorphism. Since \mathbb{R} is the Lie algebra of the simply connected Lie group \mathbb{R} , this induces a Lie group homomorphism

$$c : \mathbb{R} \longrightarrow G$$

such that

$$c(s + t) = c(s) + c(t)$$

for all s and t . The operation on the right hand side is the group multiplication in G . The formal similarity of this formula with the one valid for the exponential function justifies the definition

$$\text{Exp}(X) = c(1).$$

This is called the exponential map, and it maps the Lie algebra \mathfrak{g} into the Lie group G . It provides a diffeomorphism between a neighborhood of 0 in \mathfrak{g} and a neighborhood of Id in G . The exponential map from the Lie algebra to the Lie group is not always onto, even if the group is connected (For example, the exponential map of $sl(2, \mathbb{C})$ is not surjective.) But if \mathfrak{g} is nilpotent Exp is bijective.

1.5 Lie admissible algebras

Let \mathcal{A} be a complex associative algebra whose multiplication is denoted by $A.B$ with $A, B \in \mathcal{A}$. It is easy to see that

$$[A, B] = AB - BA$$

is a Lie bracket. Let \mathcal{A}^L the corresponding Lie algebra. For example, if $M(n, \mathbb{C})$ is the vector space of complex matrices of order n , it is an associative algebra for the usual product of matrices then $[A, B] = AB - BA$ is a Lie product on $M(n, \mathbb{C})$. This Lie algebra is usually denoted $gl(n, \mathbb{C})$. We can note that there exists Lie algebras which are not given by an associative algebra.

Definition 2 A Lie-admissible algebra is a (nonassociative) algebra \mathcal{A} whose product $A.B$ is such that

$$[A, B] = AB - BA$$

is a product of Lie algebra.

It is equivalent to say that the product $A.B$ satisfies

$$(A.B).C - A.(B.C) - (B.A).C + B.(A.C) - (A.C).B + A.(C.B) - (C.B).A + C.(B.A) + (B.C).A - B.(C.A) + (C.A).B - C.(A.C) = 0$$

for all $A, B, C \in \mathcal{A}$. An interesting example concerns the Pre-Lie algebras, that is (nonassociative) algebras whose product satisfies

$$(A.B).C - A.(B.C) - (A.C).B + A.(C.B) = 0.$$

This algebras, also called left-symmetric algebras, are studied to determinate on a Lie group the flat left invariant affine connections with torsionfree. For example, abelian Pre-Lie algebras determinates the affine structures (i.e. flat torsionfree left invariant affine connections on a Lie group) on the corresponding Lie algebra, that is the Lie algebra given by the Pre-Lie algebra. But an abelian Pre-Lie algebra is associative commutative. Let us give, for example the commutative associative algebra of dimension 2 given by

$$e_1.e_1 = e_1, e_1.e_2 = e_2.e_1 = e_2, e_2.e_2 = e_2.$$

The left translation l_X of this multiplication is given for $X = ae_1 + be_2$ by

$$l_X(e_1) = ae_1 + be_2, l_X(e_2) = (a + b)e_2.$$

The associated Lie algebra is abelian and admits a representation on the Lie algebra $Aff(\mathbb{R}^2)$ (here we are interested by the real case) given by

$$\begin{pmatrix} l_X & X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 & a \\ b & a+b & b \\ 0 & 0 & 0 \end{pmatrix}.$$

The Lie group associated to this affine Lie algebra is found taking the exponential of this matrix. We obtain

$$G = \left\{ \begin{pmatrix} e^a & 0 & e^a - 1 \\ e^a(e^b - 1) & e^a e^b & e^a(e^b - 1) \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

This gives the following affine transformations

$$\begin{cases} x \rightarrow e^a x + e^a - 1 \\ y \rightarrow e^a(e^b - 1)x + e^a e^b y + e^a(e^b - 1) \end{cases}$$

1.6 Infinite dimensional Lie algebras

The theory of infinite dimensional Lie algebras, that is the underlying vector space is of infinite dimension, is a little bit different. For example there is still no correspondance between Lie algebras and Lie groups. W.T Van Est showed example of Banach Lie algebras which are not associated to infinite Lie groups. But some infinite Lie algebras play fundamental role. The Kac-Moody algebras are graded infinite Lie algebras defined by generators and relations and which are constructed as finite dimensional simple Lie algebras. Another example is given by the Lie algebra of the vector fields of a differentiable manifold M . The bracket is given by the Lie derivative. This Lie algebra is very complicated. It is well studied in cases of $M = \mathbb{R}$ or $M = S^1$. In this case the Lie algebra is associated to the Lie group of diffeomorphisms of \mathbb{R} or S^1 . Another family of infinite Lie algebras are the Cartan Lie algebras. They are defined as Lie algebras of infinitesimal diffeomorphisms (vector fields) which leave invariant a structure as symplectic structure or contact structure. For example let (M, Ω) a symplectic variety, that is Ω is a symplectic form on the differential manifold M . We consider

$$L(M, \Omega) = \{X \text{ vector fields on } M, L_X \Omega = 0\}$$

where

$$L_X \Omega = i(X)d\Omega + d(i(X)\Omega) = d(i(X)\Omega) = 0$$

is the Lie derivative. Then $L(M, \Omega)$ is a real infinite Lie algebra. It admits a subalgebra L_0 constituted of vector fields of $L(M, \Omega)$ with compact support. Then Lichnerowicz proved that every finite dimensional Lie algebra of L_0 is reductive (that is a direct product of a semi simple sub algebra by an abelian center) and every nonzero ideal is of infinite dimensional.

2 Classifications of Lie algebras

Definition 3 Two Lie algebras \mathfrak{g} and \mathfrak{g}' of multiplication μ and $\mu' \in L^n$ are said isomorphic, if there is $f \in Gl(n, \mathbb{C})$ such that

$$\mu'(X, Y) = f * \mu(X, Y) = f^{-1}(\mu(f(X), f(Y)))$$

for all $X, Y \in \mathfrak{g}$.

For example any 2-dimensional Lie algebra is given by

$$\mu(e_1, e_2) = ae_1 + be_2.$$

Suppose a or b not zero, for example b , the change of basis

$$\begin{cases} f(e_1) = 1/b e_1 \\ f(e_2) = a/be_1 + e_2 \end{cases}$$

defines the isomorphic law

$$\mu(e_1, e_2) = e_2.$$

We deduce that every two dimensional Lie algebra is abelian or isomorphic to the Lie algebra whose product is

$$\mu(e_1, e_2) = e_2.$$

The classification of n dimensional Lie algebras consists to describe a representative Lie algebra of each class of isomorphism. Thus the classification of complex (or real) 2-dimensional Lie algebras is given by the following result :

Proposition 1 Every 2-dimensional complex (or real) Lie algebra is isomorphic to one of the following:
 - The abelian 2-dimensional Lie algebra.
 - The Lie algebra defined from $\mu(e_1, e_2) = e_2$.

The general classification, that is the classification of Lie algebra of arbitrary dimension is a very complicated problem and it is today unsolved. We know this general classification only up the dimension 5. Beyond this dimension we know only partial classifications. We are thus led to define particular classes of Lie algebras.

2.1 Simple Lie algebras

Definition 4 A Lie algebra \mathfrak{g} is called simple if it is of dimension greater or equal to 2 and do not contain proper ideal, that is ideal not trivial and not equal to \mathfrak{g} .

The classification of simple complex Lie algebras is wellknown. It is rather old and it is due primarily to works of Elie Cartan, Dynkin or Killing. To summarize this work, let us quote only the final result. Every simple complex Lie algebra is

- either of classical type that is isomorphic to one of the following: $su(n, \mathbb{C})$ (type A_n), $so(2n+1, \mathbb{C})$ (type B_n), $sp(n, \mathbb{C})$ (type C_n), $so(2n, \mathbb{C})$ (type D_n)
- or exceptional that is of type E_6, E_7, E_8, F_4, G_2 .

One can read the definition of these algebras for example in the book of J.P. Serre entitled SemiSimple Lie algebras.

Another class of Lie algebras concerns the semi-simple Lie algebra. By definition, a semi-simple Lie algebra is a direct product of simple Lie algebras. Briefly let us point out the notion of direct product. Let \mathfrak{g}_1 and \mathfrak{g}_2 two complex Lie algebras whose multiplications are denoted by μ_1 and μ_2 . The direct product $\mathfrak{g}_1 \otimes \mathfrak{g}_2$ is the Lie algebra whose underlying vectorial space is $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and whose multiplication is

$$\mu(X_1 + X_2, Y_1 + Y_2) = \mu_1(X_1, Y_1) + \mu_2(X_2, Y_2)$$

for any $X_1, Y_1 \in \mathfrak{g}_1$ and $X_2, Y_2 \in \mathfrak{g}_2$. We deduce easely, from the classification of simple Lie algebras that from semi simple complex Lie algebras.

2.2 Nilpotent Lie algebras

Let \mathfrak{g} be a Lie algebra. Let us consider the following sequence of ideals

$$\begin{cases} \mathcal{C}^1(\mathfrak{g}) = \mu(\mathfrak{g}, \mathfrak{g}) \\ \mathcal{C}^p(\mathfrak{g}) = \mu(\mathcal{C}^{p-1}(\mathfrak{g}), \mathfrak{g}) \text{ for } p > 2. \end{cases}$$

where $\mu(\mathcal{C}^{p-1}(\mathfrak{g}), \mathfrak{g})$ is the subalgebra generated by the product $\mu(X, Y)$ with $X \in \mathcal{C}^{p-1}(\mathfrak{g})$ and $Y \in \mathfrak{g}$. This sequence satisfies

$$\mathcal{C}^p(\mathfrak{g}) \subset \mathcal{C}^{p-1}(\mathfrak{g}), \quad p > 0$$

where $\mathcal{C}^0(\mathfrak{g}) = \mathfrak{g}$.

Definition 5 A Lie algebra \mathfrak{g} is called nilpotent if there is k such that

$$\mathcal{C}^k(\mathfrak{g}) = \{0\}.$$

If a such integer exists, the smallest k is called the index of nilpotency or nilindex of \mathfrak{g} .

Examples.

1. The abelian algebra satisfies $\mathcal{C}^1(\mathfrak{g}) = \{0\}$. It is nilpotent of nilindex 1.
2. The $(2p+1)$ -dimensional Heisenberg algebra is given in a basis $\{e_1, \dots, e_{2p+1}\}$ from

$$\mu(e_{2i+1}, e_{2i+2}) = e_{2p+1}$$

for $i = 0, \dots, p-1$. This Lie algebra is nilpotent of nilindex 2. In this case $\mathcal{C}^1(\mathfrak{g})$ is generated by the center $\{e_{2p+1}\}$. Generally speaking, we call 2-step nilpotent Lie algebra a nilpotent Lie algebra whose nilindex is equal to 2.

3. For any nilpotent n -dimensional Lie algebra, the nilindex is bounded by $n-1$.

Definition 6 A nilpotent Lie algebra is called filiform if its nilindex is equal to $n - 1$.

For example, the following 4-dimensional Lie algebra given by

$$\begin{cases} \mu(e_1, e_2) = e_3 \\ \mu(e_1, e_3) = e_4 \end{cases}$$

is filiform.

The classification of complex (and real) nilpotent Lie algebras is known up the dimension 7. For the dimensions 3,4,5 and 6, there exists only a finite number of classes of isomorphism. For the dimensions 7 and beyond, there exist an infinity of isomorphism classes. For example, for dimension 7, we have 6 families of one parameter of non isomorphic Lie algebras.

The classification of nilpotent Lie algebras of dimension less or equal to 7 is given in the web site

<http://www.math.uha.fr/~algebre>

2.3 Solvable Lie algebras

Let \mathfrak{g} be a Lie algebra. Let us consider the following sequence of ideals

$$\begin{cases} \mathcal{D}^1(\mathfrak{g}) = \mu(\mathfrak{g}, \mathfrak{g}) \\ \mathcal{D}^p(\mathfrak{g}) = \mu(\mathcal{D}^{p-1}(\mathfrak{g}), \mathcal{D}^{p-1}(\mathfrak{g})) \text{ for } p > 2. \end{cases}$$

This sequence satisfies

$$\mathcal{D}^p(\mathfrak{g}) \subset \mathcal{D}^{p-1}(\mathfrak{g}), \quad p > 0$$

where $\mathcal{D}^0(\mathfrak{g}) = \mathfrak{g}$.

Definition 7 A Lie algebra \mathfrak{g} is called solvable if there exists k such that

$$\mathcal{D}^k(\mathfrak{g}) = \{0\}.$$

If a such integer exists, the smallest k is called the index of solvability or solvindex of \mathfrak{g} .

Examples.

1. Any nilpotent Lie algebra is solvable because

$$\mathcal{D}^p(\mathfrak{g}) \subset \mathcal{C}^p(\mathfrak{g}).$$

2. The non abelian 2-dimensional Lie algebra is solvable.
3. We can construct solvable Lie algebra starting of a nilpotent Lie algebra. First we need to some definitions.

Definition 8 A derivation of a Lie algebra \mathfrak{g} is a linear endomorphism f satisfying

$$\mu(f(X), Y) + \mu(X, f(Y)) = f(\mu(X, Y))$$

for every $X, Y \in \mathfrak{g}$.

For example the endomorphisms adX given by

$$adX(Y) = \mu(X, Y)$$

are derivations (called inner derivations). Let us note that any inner derivation is singular because $X \in Ker(adX)$. A Lie algebra provided with a regular derivation is nilpotent. Let us note too that if \mathfrak{g} is a simple or semi simple Lie algebra, any derivation is inner.

Let \mathfrak{g} be a n -dimensional Lie algebra and f a derivation not inner. Consider the vector space $\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{C}$ of dimension $n + 1$ and we denote by e_{n+1} a basis of the complementary space \mathbb{C} . We define a multiplication on \mathfrak{g}' by

$$\begin{cases} \mu'(X, Y) = \mu(X, Y), & X, Y \in \mathfrak{g} \\ \mu'(X, e_{n+1}) = f(X), & X \in \mathfrak{g} \end{cases}$$

Then \mathfrak{g}' is a solvable Lie algebra, not nilpotent as soon as f is a non-nilpotent derivation.

3 The algebraic variety of complex Lie algebras

A n -dimensional complex Lie algebra can be seen as a pair $\mathfrak{g} = (\mathbb{C}^n, \mu)$ where μ is a Lie algebra law on \mathbb{C}^n , the underlying vector space to \mathfrak{g} is \mathbb{C}^n and μ the bracket of \mathfrak{g} . We will denote by L^n the set of Lie algebra laws on \mathbb{C}^n . It is a subset of the vectorial space of alternating bilinear mappings on \mathbb{C}^n . We recall that two laws μ and $\mu' \in L^n$ are said isomorphic, if there is $f \in Gl(n, \mathbb{C})$ such that

$$\mu'(X, Y) = f * \mu(X, Y) = f^{-1}(\mu(f(X), f(Y)))$$

for all $X, Y \in \mathbb{C}^n$. In this case, the Lie algebras $\mathfrak{g} = (\mathbb{C}^n, \mu)$ and $\mathfrak{g}' = (\mathbb{C}^n, \mu')$ are isomorphic.

We have a natural action of the linear group $Gl(n, \mathbb{C})$ on L^n given by

$$\begin{aligned} Gl(n, \mathbb{C}) \times L^n &\longrightarrow L^n \\ (f, \mu) &\longrightarrow f * \mu \end{aligned}$$

We denote by $\mathcal{O}(\mu)$ the orbit of μ respect to this action. This orbit is the set of the laws isomorphic to μ .

Notation. If \mathfrak{g} is a n -dimensional complex Lie algebra whose bracket (or law) is μ , we will denote by $\mathfrak{g} = (\mu, \mathbb{C}^n)$ this Lie algebra and often we do not distinguish the Lie algebra and its law.

3.1 The algebraic variety L^n .

Let $\mathfrak{g} = (\mu, \mathbb{C}^n)$ a n -dimensional complex Lie algebra. We fix a basis $\{e_1, e_2, \dots, e_n\}$ of \mathbb{C}^n . The structural constants of $\mu \in L^n$ are the complex numbers C_{ij}^k given by

$$\mu(e_i, e_j) = \sum_{k=1}^n C_{ij}^k e_k.$$

As the basis is fixed, we can identify the law μ with its structural constants. These constants satisfy :

$$(1) \begin{cases} C_{ij}^k = -C_{ji}^k, & 1 \leq i < j \leq n, \quad 1 \leq k \leq n \\ \sum_{l=1}^n C_{ij}^l C_{lk}^s + C_{jk}^l C_{li}^s + C_{ki}^l C_{jl}^s = 0, & 1 \leq i < j < k \leq n, \quad 1 \leq s \leq n. \end{cases}$$

Then L^n appears as an algebraic variety embedded in the linear space of alternating bilinear mapping on \mathbb{C}^n , isomorphic to $\mathbb{C}^{\frac{n^3-n^2}{2}}$.

Let be $\mu \in L^n$ and consider the Lie subgroup G_μ of $Gl(n, \mathbb{C})$ defined by

$$G_\mu = \{f \in Gl(n, \mathbb{C}) \mid f * \mu = \mu\}$$

Its Lie algebra is the Lie algebra of derivations $Der(\mathfrak{g})$ of the Lie algebra \mathfrak{g} . We can denote also this algebra by $Der(\mu)$. The orbit $\mathcal{O}(\mu)$ is isomorphic to the homogeneous space $Gl(n, \mathbb{C})/G_\mu$. Then it is a \mathcal{C}^∞ differential manifold of dimension

$$\dim \mathcal{O}(\mu) = n^2 - \dim Der(\mu).$$

Proposition 2 *The orbit $\mathcal{O}(\mu)$ of the law μ is an homogeneous differential manifold of dimension*

$$\dim \mathcal{O}(\mu) = n^2 - \dim Der(\mu).$$

3.2 The tangent space to $\mathcal{O}(\mu)$ at μ

We have seen that the orbit $\mathcal{O}(\mu)$ of μ is a differentiable manifold embedded in L^n defined by

$$\mathcal{O}(\mu) = \frac{Gl(n, \mathbb{C})}{G_\mu}$$

We consider a point μ' close to μ in $\mathcal{O}(\mu)$. There is $f \in Gl(n, \mathbb{C})$ such that $\mu' = f * \mu$. Suppose that f is close to the identity : $f = Id + \varepsilon g$, with $g \in gl(n)$. Then

$$\begin{aligned} \mu'(X, Y) &= \mu(X, Y) + \varepsilon[-g(\mu(X, Y)) + \mu(g(X), Y) + \mu(X, g(Y))] \\ &\quad + \varepsilon^2[\mu(g(X), g(Y)) - g(\mu(g(X), Y) + \mu(X, g(Y)) - g\mu(X, Y))] \end{aligned}$$

and

$$\lim_{\varepsilon \rightarrow 0} \frac{\mu'(X, Y) - \mu(X, Y)}{\varepsilon} = \delta_\mu g(X, Y)$$

where $\delta_\mu g$ defined by

$$\delta_\mu g(X, Y) = -g(\mu(X, Y)) + \mu(g(X), Y) + \mu(X, g(Y))$$

is the coboundary of the cochain g for the Chevalley cohomology of the Lie algebra \mathfrak{g} . For simplify the witting, we will denote $B^2(\mu, \mu)$ and $Z^2(\mu, \mu)$ as well as $B^2(\mathfrak{g}, \mathfrak{g})$ and $Z^2(\mathfrak{g}, \mathfrak{g})$

Proposition 3 *The tangent space to the orbit $\mathcal{O}(\mu)$ at the point μ is the space $B^2(\mu, \mu)$ of the 2-cocycles of the Chevalley cohomology of μ .*

3.3 The tangent cone to L^n at the point μ

Let μ be in L^n and consider the bilinear alternating mappings $\mu_t = \mu + t\varphi$ where t is a complex parameter. Then $\mu_t \in L^n$ for all t if and only if we have :

$$\begin{cases} \delta_\mu \varphi = 0, \\ \varphi \in L^n. \end{cases}$$

So we have the following characterisation of the tangent line of the variety L^n :

Proposition 4 *A straight line Δ passing through μ is a tangent line in μ to L^n if its direction is given by a vector of $Z^2(\mu, \mu)$.*

Suppose that $H^2(\mu, \mu) = 0$. Then the tangent space to $\mathcal{O}(\mu)$ at the point μ is the set of the tangent lines to L^n at the point μ . Thus the tangent space to L^n exists in this point and it is equal to $B^2(\mu, \mu)$. The point μ is a nonsingular point. We deduce of this that the inclusion $\mathcal{O}(\mu) \hookrightarrow L^n$ is a local homeomorphism. This property is valid for all points of $\mathcal{O}(\mu)$, then $\mathcal{O}(\mu)$ is open in L^n (for the induced metric topology).

Proposition 5 *Let $\mu \in L^n$ such that $H^2(\mu, \mu) = 0$. If the algebraic variety L^n is provided with the metric topology induced by $\mathbb{C}^{\frac{n^3-n^2}{2}}$, then the orbit $\mathcal{O}(\mu)$ is open in L^n .*

This geometrical approach shows the significance of problems underlying to the existence of singular points in the algebraic variety L^n . We are naturally conducted to use the algebraic notion of the scheme associated to L^n . Recall that the notions of algebraic variety and the corresponding scheme coincide if this last is reduced. In the case of L^n , we will see that the corresponding scheme is not reduced.

4 The scheme \mathcal{L}^n .

4.1 Definition

Consider the formal variables C_{jk}^i with $1 \leq i < j \leq n$ and $1 \leq k \leq n$ and let us note $\mathbb{C}[C_{jk}^i]$ the ring of polynomials in the variables C_{jk}^i . Let I_J the ideal of $\mathbb{C}[C_{jk}^i]$ generated by the polynomials associated to the Jacobi relations :

$$\sum_{l=1}^n C_{ij}^l C_{lk}^s + C_{jk}^l C_{li}^s + C_{ki}^l C_{jl}^s$$

with $1 \leq i < j < k \leq n$, and $1 \leq s \leq n$. The algebraic variety L^n is the algebraic set associated to the ideal I_J :

$$L^n = V(I_J)$$

We will note, for the Jacobi ideal I_J of $\mathbb{C}[C_{jk}^i]$

$$\text{rad}I_J = \{P \in \mathbb{C}[C_{jk}^i], \text{ such that } \exists r \in \mathbb{N}, P^r \in I_J\}$$

In general, $radI_J \neq I_J$ (Recall that if I is a maximal ideal, then $radI = I$. It is also the case when I is a prime ideal). If M is a subset of $\mathbb{C}^{\frac{n^3-n^2}{2}}$, we note by $\mathfrak{i}(M)$ the ideal of $\mathbb{C}[C_{jk}^i]$ defined by

$$\mathfrak{i}(M) = \{P \in \mathbb{C}[C_{jk}^i], P(x) = 0 \quad \forall x \in M\}.$$

Then we have

$$\mathfrak{i}(L^n) = \mathfrak{i}(V(I_J)) = radI_J$$

We consider the ring

$$A(L^n) = \frac{\mathbb{C}[C_{jk}^i]}{I_J}$$

which also is a finite type \mathbb{C} -algebra. This algebra corresponds to the ring of regular functions on L^n . Recall that an ideal I of a ring A is a prime ideal if the quotient ring A/I is an integral ring. In particular, maximal ideals are prime ideals. The quotient ring is called reduced if it doesn't contain nonnull nilpotent element ($\forall P \neq 0, \forall n, P^n \neq 0$). As we have generally $radI_J \neq I_J$, the algebra $A(L^n)$ is not reduced.

The affine algebra $\Gamma(L^n)$ of the algebraic variety L^n is the quotient ring

$$\Gamma(L^n) = \frac{\mathbb{C}[C_{jk}^i]}{\mathfrak{i}(L^n)}.$$

As we have the following inclusion

$$\mathfrak{i}(L^n) = radi(L^n),$$

we deduce that the ring $\Gamma(L^n)$ always is reduced.

Let us note $Spm(A(L^n))$ the set of maximal ideals of the algebra $A(L^n)$. We have a natural bijection between $Spm(A(L^n))$ and L^n :

$$Spm(A(L^n)) \sim L^n.$$

We provide $Spm(A(L^n))$ with the Zarisky topology : Let \mathfrak{a} be an ideal of $A(L^n)$ and we consider the set $V(\mathfrak{a})$ of maximal ideals of $A(L^n)$ containing \mathfrak{a} (in fact we can suppose that $V(\mathfrak{a})$ is the set of radical ideals containing \mathfrak{a}). These sets are the closed sets of the topology of $Spm(A(L^n))$. We define a basis of open sets considering, for $f \in A(L^n)$:

$$D(f) = \{x \in Spm(A(L^n)), f(x) \neq 0\}.$$

There exists a sheaf of functions $\mathcal{O}_{Spm(A(L^n))}$ on $Spm(A(L^n))$ with values on \mathbb{C} such that, for all $f \in A(L^n)$, we have

$$\Gamma(D(f), \mathcal{O}_{Spm(A(L^n))}) = A(L^n)_f$$

where $A(L^n)_f$ is the ring of functions

$$x \rightarrow \frac{g(x)}{f(x)^n}$$

for $x \in D(f)$ and $g \in A(L^n)$. In particular we have

$$\Gamma(Spm(A(L^n)), \mathcal{O}_{Spm(A(L^n))}) = A(L^n)$$

The affine scheme of ring $A(L^n)$ is the space $(Spm(A(L^n)), \mathcal{O}_{Spm(A(L^n))})$ noted also $Spm(A(L^n))$ or \mathcal{L}^n .

4.2 The tangent space of the scheme \mathcal{L}^n

The tangent space to the scheme $Spm(A(L^n))$ can be calculated classically. We consider the infinitesimal deformations of the algebra $\Gamma(Spm(A(L^n)), \mathcal{O}_{Spm(A(L^n))})$ at a given point. If F_1, \dots, F_N with $N = \frac{1}{6}n(n-1)(n-2)$ are the Jacobi polynomials, then the tangent space at the point x to the scheme \mathcal{L}^n is

$$T_x(\mathcal{L}^n) = Ker d_x(F_1, \dots, F_N)$$

where d_x designates the Jacobian matrix of the F_i at the point x . From the definition of the cohomology of the Lie algebra \mathfrak{g} associated to the point x , we have :

Theorem 1 $T_x(Spm(A(L^n))) = T_x(\mathcal{L}^n) = Z^2(\mathfrak{g}, \mathfrak{g})$

5 Contractions of Lie algebras

Sometimes the word "contraction" is replaced by the word "degeneration". But we prefer here use the term contraction because its sense is more explicit.

Let us consider the complex algebraic variety L^n . This variety can be endowed with the Zariski topology (the closed set are defined by a finite number of polynomial equations on the parameters C_{ij}^k .) It can be also endowed with the metric topology induced by \mathbb{C}^N where $N = \frac{n^3-n^2}{2}$ the vector space of structural constants, considering the embedding $L^n \in \mathbb{C}^N$. Recall that every open set for the Zariski topology is an open set for the metric topology. If A is a subset of L^n , we note \bar{A} the closure of A in L^n for the Zariski topology and \bar{A}^d its closure for the metric topology.

5.1 Definition of contractions of Lie algebras

Let $g = (\mu, \mathbb{C}^n)$ be a n -dimensional complex Lie algebra.

Definition 9 A Lie algebra $\mathfrak{g}_0 = (\mu_0, \mathbb{C}^n)$, $\mu_0 \in L^n$ is called contraction of \mathfrak{g} if $\mu_0 \in \overline{\mathcal{O}(\mu)}$.

The historical notion of contraction, given by Segal, was the following. Consider a sequence $\{f_p\}$ in $Gl(n, \mathbb{C})$. We deduce a sequence $\{\mu_p\}$ in L^n by putting

$$\mu_p = f_p * \mu.$$

If this sequence admits a limit μ_0 in \mathbb{C}^N , then $\mu_0 \in L^n$ and μ_0 was called a contraction of μ . The link between these two notions of contraction is given on the following proposition.

Proposition 6 For every $\mu \in L^n$, the Zariski closure $\overline{\mathcal{O}(\mu)}$ of the orbit $\mathcal{O}(\mu)$ is equal to the metric closure $\overline{\mathcal{O}(\mu)}^d$:

$$\overline{\mathcal{O}(\mu)} = \overline{\mathcal{O}(\mu)}^d.$$

Proof. In fact the field of coefficients is \mathbb{C} .

Corollary 1 *Every contraction of $\mu \in L^n$ is obtained by a Segal contraction.*

5.2 Examples

5.2.1 The abelian case

Every Lie algebra can be contracted on the abelian Lie algebra. In fact if the law μ is defined on a basis $\{X_i\}$ by $\mu(X_i, X_j) = \sum C_{ij}^k X_k$, we consider the isomorphism $f_\varepsilon(X_i) = \varepsilon X_i, \varepsilon \neq 0$. Then the law $\mu_\varepsilon = f_\varepsilon * \mu$ satisfies $\mu_\varepsilon(X_i, X_j) = \sum \varepsilon C_{ij}^k X_k$ and $\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon$ exists and coincides with the law of the abelian Lie algebra.

5.2.2 Contact Lie algebras

Let us consider the open set \mathcal{C}_{2p+1} of L^{2p+1} constituted of $(2p+1)$ -dimensional Lie algebra endowed with a contact form, that is $\omega \in \mathfrak{g}^*$ (the dual of \mathfrak{g}) satisfying

$$\omega \wedge (d\omega)^p \neq 0.$$

There is a basis $(X_1, X_2, \dots, X_{2p+1})$ of \mathfrak{g} such that the dual basis $(\omega = \omega_1, \omega_2, \dots, \omega_{2p+1})$ satisfies

$$d\omega_1 = \omega_2 \wedge \omega_3 + \omega_4 \wedge \omega_5 + \dots + \omega_{2p} \wedge \omega_{2p+1}.$$

The structural constants respect this basis have the form

$$C_{23}^1 = C_{34}^1 = \dots = C_{2p2p+1}^1 = 1.$$

Consider the isomorphism of \mathbb{C}^{2p+1} given by

$$f_\varepsilon(X_1) = \varepsilon^2 X_1, \quad f_\varepsilon(X_i) = \varepsilon X_i \quad i = 2, \dots, 2p+1.$$

The structural constant D_{ij}^k of $\mu_\varepsilon = f_\varepsilon * \mu$ respect the basis $\{X_i\}$ satisfy

$$\begin{cases} D_{23}^1 = D_{34}^1 = \dots = D_{2p2p+1}^1 = 1 \\ D_{ij}^k = \varepsilon C_{ij}^k \text{ for others indices} \end{cases}$$

This implies that $\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon$ exists and corresponds to the law \mathfrak{h}_p of the Heisenberg algebra of dimension $2p+1$. Then $\mathfrak{h}_p \in \overline{\mathcal{C}_{2p+1}}$.

Proposition 7 *Every $2p+1$ -dimensional Lie algebra provided with a contact form can be contracted on the $2p+1$ -dimensional Heisenberg algebra \mathfrak{h}_p . Moreover, every Lie algebra which is contracted on \mathfrak{h}_p admits a contact form.*

5.2.3 Frobeniusian Lie algebras

Let \mathfrak{g} be a $2p$ -dimensional Lie algebra. It called frobeniusian if there exists a non trivial linear form $\omega \in \mathfrak{g}^*$ such that $[d\omega]^p \neq 0$. In this case the 2-form $\theta = d\omega$ is an exact symplectic form on \mathfrak{g} .

Theorem 2 [?] *Let $\{F_\varphi \mid \varphi \in \mathbb{C}^{p-1}\}$ be the family on $(p-1)$ -parameters of $2p$ -dimensional Lie algebras given by*

$$\begin{cases} d\omega_1 = \omega_1 \wedge \omega_2 + \sum_{k=1}^{p-1} \omega_{2k+1} \wedge \omega_{2k+2} \\ d\omega_2 = 0 \\ d\omega_{2k+1} = \varphi_k \omega_2 \wedge \omega_{2k+1}, \quad 1 \leq k \leq p-1 \\ d\omega_{2k+2} = -(1 + \varphi_k) \omega_2 \wedge \omega_{2k+2}, \quad 1 \leq k \leq p-1 \end{cases}$$

where $\{\omega_1, \dots, \omega_{2p}\}$ is a basis of $(\mathbb{C}^{2p})^*$. The family $\{F_\varphi\}$ is a complex irreducible multiple model for the property "there exists a linear form whose differential is symplectic", that is every $2p$ -dimensional frobeniusian complex Lie algebra can be contracted on a Lie algebra belonging to the family $\{F_\varphi\}$.

It can be easily seen that the algebras F_φ admit the following graduation: if $\{X_1, \dots, X_{2p}\}$ is a dual basis to $\{\omega_1, \dots, \omega_{2p}\}$, then $F_\varphi = (F_\varphi)_0 \oplus (F_\varphi)_1 \oplus (F_\varphi)_2$, where $(F_\varphi)_0 = \mathbb{C}X_2$, $(F_\varphi)_1 = \sum_{k=3}^{2p} \mathbb{C}X_k$ and $(F_\varphi)_2 = \mathbb{C}X_1$. This decomposition will be of importance for cohomological computations.

Proof. Let \mathfrak{g} be a $2p$ -dimensional complex frobeniusian Lie algebra. There exists a basis $\{X_1, \dots, X_{2p}\}$ of \mathfrak{g} such that the dual basis $\{\omega_1, \dots, \omega_{2p}\}$ satisfies

$$d\omega_1 = \omega_1 \wedge \omega_2 + \dots + \omega_{2p-1} \wedge \omega_{2p},$$

i.e. the linear form ω_1 being supposed frobeniusian. Let us consider the one parameter change of basis :

$$f_\epsilon(X_1) = \epsilon^2 X_1, \quad f_\epsilon(X_2) = X_2, \quad f_\epsilon(X_i) = \epsilon X_i, \quad i = 3, \dots, 2p.$$

This family defines, when $\epsilon \rightarrow 0$, a contraction \mathfrak{g}_0 of \mathfrak{g} whose Cartan Maurer equations are

$$\begin{cases} d\omega_1 = \omega_1 \wedge \omega_2 + \dots + \omega_{2p-1} \wedge \omega_{2p}, \\ d\omega_2 = 0, \\ d\omega_3 = C_{23}^3 \omega_2 \wedge \omega_3 + C_{24}^3 \omega_2 \wedge \omega_4 + \dots + C_{22p-1}^3 \omega_2 \wedge \omega_{2p-1} + C_{22p}^3 \omega_2 \wedge \omega_{2p}, \\ d\omega_4 = C_{23}^4 \omega_2 \wedge \omega_3 + (-1 - C_{23}^3) \omega_2 \wedge \omega_4 + \dots + C_{22p-1}^4 \omega_2 \wedge \omega_{2p-1} + C_{22p}^4 \omega_2 \wedge \omega_{2p}, \\ \dots \\ d\omega_{2p-1} = C_{22p}^4 \omega_2 \wedge \omega_3 + C_{2p}^3 \omega_2 \wedge \omega_4 + \dots + C_{22p-1}^{2p-1} \omega_2 \wedge \omega_{2p-1} + C_{22p}^{2p-1} \omega_2 \wedge \omega_{2p}, \\ d\omega_{2p} = C_{22p-1}^4 \omega_2 \wedge \omega_3 + C_{22p-1}^3 \omega_2 \wedge \omega_4 + \dots + C_{22p-1}^{2p} \omega_2 \wedge \omega_{2p-1} + (-1 - C_{22p-1}^{2p-1}) \omega_2 \wedge \omega_{2p}. \end{cases}$$

The end of the proof consists to reduce the operator ψ which is defined as the restriction of the adjoint operator adX_2 to the invariant linear subspace F generated by $\{X_3, \dots, X_{2p}\}$. We can directly verify the following sentences :

- If α and β are eigenvalues of ψ such that $\alpha \neq -1 - \beta$, then the eigenspaces F_α and F_β satisfy $[F_\alpha, F_\beta] = 0$.
- If the eigenvalue α of ψ us tot equal to $-\frac{1}{2}$, then, for every X and $Y \in F_\alpha$, we have $[X, Y] = 0$.
- If α is an eigenvalue α of ψ , then $-1 - \alpha$ is also an eigenvalue of ψ .

- The multiplicities of the eigenvalues α and $-1 - \alpha$ are equal.
- The ordered sequences of Jordan blocks corresponding to the eigenvalues α and $-1 - \alpha$ are the same.

From these remarks, we can find a Jordan basis of ψ such that the matrix of ψ restricted to the invariant subspace $C_\alpha \oplus C_{-1-\alpha}$ where C_λ designates the characteristic subspaces associated to the eigenvalue λ has the following form :

$$\begin{pmatrix} \alpha & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & -1 - \alpha & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & \alpha & 0 & 1 & 0 & \dots \\ 0 & -1 & 0 & -1 - \alpha & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \alpha & 0 & \dots \\ 0 & 0 & 0 & -1 & 0 & -1 - \alpha & \dots \end{pmatrix}.$$

Then the eigenvalues and their corresponding previous blocks classify the elements of the family of frobeniusian models.

We can find also real models for real frobeniusian algebras. In this case the previous theorem is written

Theorem 3 [?] (real case). Let $\{F_{\varphi,\rho} \mid \varphi, \rho \in \mathbb{R}^{p-1}\}$ be the family on $(2p-2)$ -parameters of $2p$ -dimensional real Lie algebras given by

$$\left\{ \begin{array}{l} [X_1, X_2] = X_1, \\ [X_{2k+1}, X_{2k+2}] = X_1, \quad k = 0, \dots, p-1, \\ [X_2, X_{4k-1}] = \varphi_k X_{4k-1} + \rho_k X_{4k+1}, \\ [X_2, X_{4k}] = (-1 - \varphi_k) X_{4k} - \rho_k X_{4k+2}, \\ [X_2, X_{4k+1}] = \rho_k X_{4k-1} + \varphi_k X_{4k+1}, \\ [X_2, X_{4k+2}] = \rho_k X_{4k} + (-1 - \varphi_k) X_{4k+2}, \\ \\ \text{for every } k \leq s, \\ \\ [X_2, X_{4s+2k-1}] = -\frac{1}{2} X_{4s+2k-1} + \rho_{k+s-1} X_{4s+2k}, \\ [X_2, X_{4s+2k}] = -\rho_{k+s-1} X_{4s+2k-1} + -\frac{1}{2} X_{4s+2k}, \\ \\ \text{for every } 2 \leq k \leq p-2 \end{array} \right.$$

where s is a parameter satisfying $0 \leq \lfloor \frac{p-1}{2} \rfloor$. The family $\{F_{\varphi,\rho}\}$ is a real irreducible multiple model for the property "there exists a linear form whose differential is symplectic", that is every $2p$ -dimensional frobeniusian real Lie algebra can be contracted on a Lie algebra belonging to the family $\{F_\varphi\}$.

5.3 Inönü-Wigner contractions

The first concept of contractions of Lie algebras has been introduced by Segal, Inönü and Wigner for explain some properties related with he classical mechanics, the relativist mechanic and the quantum mechanic. The basic idea is to joint these two last theories with the classical or the galilean mechanic

when the fundamental constants (light velocity, Planck constant) tends to infinity or zero. In this context, a contraction is written $\lim_{n \rightarrow \infty} f_n^{-1}[f_n(X), f_n(Y)]$ if this limit exists. The Inönü-Wigner contractions are a particular case of these limits. We consider a family of isomorphisms $\{f_\epsilon\}$ in $Gl(n, \mathbb{C})$ of the form

$$f_\epsilon = f_1 + \epsilon f_2$$

where $f_1 \in gl(n, \mathbb{C})$ satisfying is a singular operator $\det(f_1) = 0$ and $f_2 \in Gl(n, \mathbb{C})$. These endomorphisms can be easily reduced to the following form:

$$f_1 = \begin{pmatrix} Id_r & 0 \\ 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} v & 0 \\ 0 & Id_{n-r} \end{pmatrix}$$

with $\text{rank}(f_1) = \text{rank}(v) = r$.

Such contractions permits to contracts a given Lie algebra \mathfrak{g} on a Lie algebra \mathfrak{g}_0 by staying invariant a subalgebra \mathfrak{h} of \mathfrak{g} that is \mathfrak{h} is always a subalgebra of \mathfrak{g}_0 . For example, the homogeneous Lorentz algebra can be contracted, via an Inonu-Wigner contraction, on the homogeneous Galilean algebra. In the same way, the De Sitter algebras can be contracted on the non-homogeneous Lorentz algebra. Now we give a short description of the Inonu-Wigner contractions. Let $\mathfrak{g} = (\mu, \mathbb{C}^n)$ be a Lie algebra and let \mathfrak{h} a Lie subalgebra of \mathfrak{g} . Suppose that $\{e_1, \dots, e_n\}$ is the fixed basis of \mathbb{C}^n and $\{e_1, \dots, e_p\}$ is a basis of \mathfrak{h} . Thus

$$\mu(e_i, e_j) = \sum_{k=1}^p C_{ij}^k e_k, \quad i, j = 1, \dots, p.$$

Let us consider the Inönü-Wigner isomorphisms given by

$$\begin{cases} f_\epsilon(e_i)(1 + \epsilon)e_i, & i = 1, \dots, p \\ f_\epsilon(e_l) = \epsilon e_l, & l = p + 1, \dots, n. \end{cases}$$

Here we have $f_\epsilon = f_1 + \epsilon f_2$ with

$$f_1 = \begin{pmatrix} Id_p & 0 \\ 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} Id_p & 0 \\ 0 & Id_{n-p} \end{pmatrix}.$$

The multiplication $\mu_\epsilon = f_\epsilon * \mu$ is written

$$\begin{cases} \mu_\epsilon(e_i, e_j) = (1 + \epsilon)^{-1} \mu(e_i, e_j), & i, j = 1, \dots, p \\ \mu_\epsilon(e_i, e_l) = \epsilon(1 + \epsilon)^{-1} \sum_{k=1}^p C_{ij}^k e_k + (1 + \epsilon)^{-1} \sum_{k=p+1}^n C_{il}^k e_k, & i = 1, \dots, p, l = p + 1, \dots, n \\ \mu_\epsilon(e_l, e_m) = \epsilon^2(1 + \epsilon)^{-1} \sum_{k=1}^p C_{lm}^k e_k + \epsilon \sum_{k=p+1}^n C_{lm}^k e_k, & l, m = p + 1, \dots, n \end{cases}$$

If $\epsilon \rightarrow 0$, the sequence $\{\mu_\epsilon\}$ has a limit μ_0 given by

$$\begin{cases} \mu_0(e_i, e_j) = \mu(e_i, e_j), & i, j = 1, \dots, p \\ \mu_0(e_i, e_l) = \sum_{k=p+1}^n C_{il}^k e_k, & i = 1, \dots, p, l = p + 1, \dots, n \\ \mu_0(e_l, e_m) = 0, & l, m = p + 1, \dots, n. \end{cases}$$

The Lie algebra $\mathfrak{g}_0 = (\mu_0, \mathbb{C}^n)$ is an Inonu-Wigner contraction of \mathfrak{g} and \mathfrak{h} is a subalgebra of \mathfrak{g}_0 . We note that the subspace $\mathbb{C}\{e_{p+1}, \dots, e_n\}$ is an abelian subalgebra of \mathfrak{g}_0 .

Proposition 8 *If \mathfrak{g}_0 is an Inönü-Wigner contraction of \mathfrak{g} which led invariant the subalgebra \mathfrak{h} of \mathfrak{g} then*

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{a}$$

where \mathfrak{a} is an abelian ideal of \mathfrak{g}_0 .

Remarks

1. If \mathfrak{h} is an ideal of \mathfrak{g} then $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{a}$ with

$$[\mathfrak{h}, \mathfrak{a}] = 0.$$

2. Later Saletan and Levy-Nahas have generalized the notion of Inönü-Wigner contractions considering
 - a) For Saletan contractions the isomorphisms

$$f_\varepsilon = f_1 + \varepsilon f_2$$

with $\det(f_2) \neq 0$. Such an isomorphism can be written

$$f_\varepsilon = \varepsilon Id + (1 - \varepsilon)g$$

with $\det(g) = 0$. If q is the nilindex of the nilpotent part of g in its Jordan decomposition, then starting with a Lie algebra \mathfrak{g} we contract via f_ε we obtain a new Lie algebra \mathfrak{g}_1 , we contract \mathfrak{g}_1 via f_ε and we obtain a Lie algebra \mathfrak{g}_2 and so on. Thus we construct a sequence of contractions. This sequence is stationary from the order q . The Inönü-Wigner case corresponds to $q = 1$.

b) Levi-Nahas extends the notion of Saletan contractions considering singular contractions. In this case the isomorphism f has the following form

$$f = \varepsilon f_1 + (\varepsilon)^2 f_2,$$

f_1 and f_2 satisfying the Saletan hypothesis.

3. In his book, R. Hermann introduce also a notion of contraction but this notion does not correspond to our definition. In fact he considers some singular contractions where the dimension is not invariant. In our presentation the dimension of \mathfrak{g} and its contracted are the same. In the Hermann definition, this is not the case.

4. There exists contraction of Lie algebras which are not Inönü-Wigner contractions. For example consider the 4-dimensional solvable Lie algebra given by

$$[e_1, e_2] = e_2, [e_3, e_4] = e_4.$$

This Lie algebra can be contracted on the nilpotent filiform Lie algebra :

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4.$$

From the previous proposition, this contraction cannot be an Inonu-Wigner contraction.

5.4 Inönü-Wigner contractions of Lie groups

There exists a notion of Inönü-Wigner contraction of Lie groups. It is subordinated to the contraction of its Lie algebras. Then every Lie group can be contracted in the Inönü-Wigner sense of its one parameter subgroup. The three dimensional rotation group is contracted to the Euclidean group of two dimension. Contraction of the homogeneous Lorentz group with respect to the subgroup which leaves invariant the coordinate temporal yields the homogeneous Galilei group. Contraction of the inhomogeneous Lorentz group with respect to the subgroup generated by the spatial rotations and the time displacements yields the full Galilei group. Contraction of the De Sitter group yields the inhomogeneous Lorentz group. All these exemples are described in the historical paper of Inönü-Wigner [17].

5.5 Weimar-Woods contractions

These contractions are given by diagonal isomrphisms

$$f(e_i) = \epsilon^{n_i} e_i$$

where $n_i \in \mathbb{Z}$ and the contraction is given when $\epsilon \rightarrow 0$. This contraction ca be viewed as generalized Inönü-Wigner contraction with integer exponents and the aim is to construct all the possible contractions. In the following section we will present the notion of deformation. We will see that any contraction of a Lie algebra \mathfrak{g} on \mathfrak{g}_1 determinates a deformation of \mathfrak{g}_1 isomorphic to \mathfrak{g}_0 . The notion of Weimer-Woods contractions permits to solve the reciprocity.

5.6 The diagrams of contractions

In the following the symbol $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ means that \mathfrak{g}_1 can be contracted on \mathfrak{g}_2 and there is not Lie algebra \mathfrak{g} such that \mathfrak{g}_1 can be contracted on \mathfrak{g} and \mathfrak{g} on \mathfrak{g}_3 (there is no intermediaire).

The two-dimensional case :

The variety L^2 is the union of 2 orbits :

$$L^2 = \mathcal{O}(\mu_0^2) \cup \mathcal{O}(\mu_1^2)$$

where μ_0^2 is the law of the 2-dimensional abelian Lie algebra and μ_1^2 given by

$$\mu_1^2(e_1, e_2) = e_2.$$

We have

$$L^2 = \overline{\mathcal{O}(\mu_1^2)}.$$

Let \mathfrak{a}_2 the abelian Lie albegra and \mathfrak{t}_2 the solvable Lie algebra of law μ_1^2 . The diagramm of contraction is

$$\mathfrak{t}_2 \longrightarrow \mathfrak{a}_2.$$

6 Formal deformations

6.1 The Gerstenhaber products

Let V be the n -dimensional \mathbb{C} -vector space \mathbb{C}^n . We denote by $V^{\otimes p}$ the tensor product of V with itself p times and $V_p = Hom_{\mathbb{C}}(V^{\otimes p+1}, V) = Bil(V \times V \times \dots \times V, V)$ the \mathbb{C} -vector space of $(p+1)$ -linear forms on V with values on V . We have $V_0 = End(V)$ and we put $V_{-1} = V$ and $V_p = 0$ for $p < -1$. We obtain a sequence $(V_p)_{p \in \mathbb{Z}}$ of \mathbb{C} -vector spaces and we note $\mathcal{V} = \bigoplus_{p \in \mathbb{Z}} V_p$. We define products noted \circ_i on \mathcal{V} putting :

if $\phi \in V_p, \psi \in V_q$ then $\phi \circ_i \psi \in V_{p+q}$ by setting

$$\begin{aligned} \phi \circ_i \psi(v_0 \otimes \dots \otimes v_{i-1} \otimes w_0 \otimes \dots \otimes w_q \otimes v_{i+1} \otimes \dots \otimes v_p) \\ = \phi(v_0 \otimes \dots \otimes v_{i-1} \otimes \psi(w_0 \otimes \dots \otimes w_q) \otimes v_{i+1} \otimes \dots \otimes v_p). \end{aligned}$$

The system $\{V_m, \circ_i\}$ is a right pre-Lie system, that is we have the following properties where $\phi = \phi^p$ to indicate its degree :

$$(\phi^p \circ_i \psi^q) \circ_j \rho^r = \begin{cases} (\phi^p \circ_j \rho^r) \circ_{i+r} \psi^q & \text{if } 0 \leq j \leq i-1 \\ \phi^p \circ_i (\psi^q) \circ_{j-i} \rho^r & \text{if } i \leq j \leq q+1, \end{cases}$$

where $\phi, \psi, \rho \in V_p, V_q, V_r$ respectively. We now define for every p and q a new homomorphism \circ of $V_p \otimes V_q$ into V_{p+q} by setting for $\phi \in V_p, \psi \in V_q$

$$\phi \circ \psi = \begin{cases} \phi \circ_0 \psi + \phi \circ_1 \psi + \dots + \phi \circ_p \psi & \text{if } q \text{ is even} \\ \phi \circ_0 \psi - \phi \circ_1 \psi + \dots + (-1)^p \phi \circ_p \psi & \text{if } q \text{ is odd} \end{cases}$$

The vector space $\mathcal{V} = \bigoplus V_p$ is , endowed with the product \circ , a graded pre-Lie algebra. That is, the graded associator satisfies:

$$(\phi^p \circ \psi^q) \circ \rho^r - (-1)^{pq} (\phi^p \circ \rho^r) \circ \psi^q = \phi^p \circ (\psi^q \circ \rho^r) - (-1)^{pq} \phi^p \circ (\rho^r) \circ \psi^q.$$

Example Let be μ a bilinear mapping on \mathbb{C}^n with values into \mathbb{C} . Then

$$\mu \circ \mu(X, Y, Z) = \mu(\mu(X, Y), Z) - \mu(X, \mu(Y, Z))$$

and we obtain the associator of the multiplication μ . This example shows that the class of associative algebras is related by the equation

$$\mu \circ \mu = 0.$$

Remark.

Starting from the Gerstenhaber products, E.Remm [14] introduces some new products directly related with some non associative algebras. Let Σ_n be the symmetric group corresponding to the permutations of n elements and $\Sigma_{p+1, q+1}$ with $p+1+q+1=n$ the subgroup of the $(p+1, q+1)$ -shuffles. Let us consider a subgroup G of $\Sigma_{p+1, q+1}$. The Remm product related with G is given by

$$(\phi^p \circ_G \psi^q) = \sum_{\sigma \in G} (\phi^p \circ \psi^q) \cdot \sigma$$

with $\sigma(v_0, \dots, v_i, w_0, \dots, w_q, v_{i+1}, \dots, v_p) = (-1)^{\epsilon(\sigma)}(v_{\sigma_1(0)}, \dots, v_{\sigma_1(i)}, w_{\sigma_2(0)}, \dots, w_{\sigma_2(q)}, v_{\sigma_1(i+1)}, \dots, v_{\sigma_1(p)})$ where $\sigma = (\sigma_1, \sigma_2)$ and $\sigma_1 \in \Sigma_{p+1}$ and $\sigma_2 \in \Sigma_{q+1}$.

The most interesting application concerning these products is in the definition of some non-associative algebras. For this consider a multiplication on \mathbb{C}^2 , that is $\mu \in V_1$. Let Σ_3 be the symmetric group corresponding to the permutations of three elements. This group contains 6 subgroups which are :

- . $G_1 = \{Id\}$
- . $G_2 = \{Id, \tau_{12}\}$
- . $G_3 = \{Id, \tau_{23}\}$
- . $G_4 = \{Id, \tau_{13}\}$
- . $G_5 = A_3$ the alternated group
- . $G_6 = \Sigma_3$

where τ_{ij} designates the transposition between i and j . For each one of these subgroups we define the following non-associative algebra given by the equation:

$$\mu \circ_{G_i} \mu = 0.$$

Of course, for $i = 1$ we obtain an associative multiplication ($\circ_{G_1} = \circ$). For $i = 2$ we obtain the class of Vinberg algebras, for $i = 3$ the class of pre-Lie algebras and for $i = 6$ the general class of Lie-admissible algebras. The Jacobi condition corresponds to $i = 5$. A large study of these product is made in [?], [?].

For end this section, we introduce another notation : let be ϕ and ψ in V_1 , and suppose that these bilinear mapping are alternated. In this case we write

$$\varphi \circ \psi = (1/2)\varphi \circ_{G_5} \psi.$$

Thus

$$\varphi \circ \psi(X, Y, Z) = \varphi(\psi(X, Y), Z) + \varphi(\psi(Y, Z), X) + \varphi(\psi(Z, X), Y)$$

for all $X, Y, Z \in \mathbb{C}^n$. Using this notation, the Lie bracket is written $\mu \circ \mu = 0$.

Application.

Let be $\mu_0 \in L^n$ a Lie algebra law and $\varphi \in C^2(\mathbb{C}^n, \mathbb{C}^n)$ that is a skew-symmetric mapping belonging to V_1 . Then $\varphi \in Z^2(\mu_0, \mu_0)$ if and only if

$$\mu_0 \circ \varphi + \varphi \circ \mu_0 = \delta_{\mu_0} \varphi = 0.$$

6.2 Formal deformations

Definition 10 A (formal) deformation of a law $\mu_0 \in L^n$ is a formal sequence with parameter t

$$\mu_t = \mu_0 + \sum_{t=1}^{\infty} t^i \varphi_i$$

where the φ_i are skew-symmetric bilinear maps $\mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that μ_t satisfies the formal Jacobi identity $\mu_t \circ \mu_t = 0$.

Let us develop this last equation.

$$\mu_t \circ \mu_t = \mu_0 \circ \mu_0 + t\delta_{\mu_0} \varphi_1 + t^2(\varphi_1 \circ \varphi_1 + \delta_{\mu_0} \varphi_2) + t^3(\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 + \delta_{\mu_0} \varphi_3) + \dots$$

and the formal equation $\mu_t \circ \mu_t = 0$ is equivalent to the infinite system

$$(I) \left\{ \begin{array}{l} \mu_0 \circ \mu_0 = 0 \\ \delta_{\mu_0} \varphi_1 = 0 \\ \varphi_1 \circ \varphi_1 = -\delta_{\mu_0} \varphi_2 \\ \varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 = -\delta_{\mu_0} \varphi_3 \\ \vdots \\ \varphi_p \circ \varphi_p + \sum_{1 \leq i \leq p-1} \varphi_i \circ \varphi_{2p-i} + \varphi_{2p-i} \circ \varphi_i = -\delta_{\mu_0} \varphi_{2p} \\ \sum_{1 \leq i \leq p} \varphi_i \circ \varphi_{2p+1-i} + \varphi_{2p+1-i} \circ \varphi_i = -\delta_{\mu_0} \varphi_{2p+1} \\ \vdots \end{array} \right.$$

Then the first term φ_1 of a deformation μ_t of a Lie algebra law μ_0 belongs to $Z^2(\mu_0, \mu_0)$. This term is called the infinitesimal part of the deformation μ_t of μ_0 .

Definition 11 A formal deformation of μ_0 is called linear deformation if it is of length one, that is of the type $\mu_0 + t\varphi_1$ with $\varphi_1 \in Z^2(\mu_0, \mu_0)$.

For a such deformation we have necessarily $\varphi_1 \circ \varphi_1 = 0$ that is $\varphi_1 \in L^n$.

Examples

1. Let μ_0 be the law of the n -dimensional nilpotent Lie algebra defined by

$$[X_1, X_i] = X_{i+1}$$

for $i = 2, \dots, n-1$, the other brackets being equal to 0. Consider a filiform n -dimensional Lie algebra law μ , that is a nilpotent Lie algebra of which nilindex is equal to $n-1$. Then we prove [14] that μ is isomorphic to an infinitesimal formal deformation of μ_0 .

2. Let us consider a $2p$ -dimensional frobeniusian complex Lie algebra. In the previous section we have done the classification of these algebras up a contraction. In [5] we prove that every $2p$ -dimensional frobeniusian complex Lie algebra law can be written, up an isomorphism, as

$$\mu = \mu_0 + t\varphi_1$$

where μ_0 is one of a model laws. This prove that every $2p$ -dimensional frobeniusian complex Lie algebra is a linear deformation of a model.

Now consider $\varphi_1 \in Z^2(\mu_0, \mu_0)$ for $\mu_0 \in L^n$. It is the infinitesimal part of a formal deformation of μ_0 if and only if there are $\varphi_i \in C^2(\mu_0, \mu_0)$, $i \geq 2$, such that the system (I) is satisfied. This existence problem is called the *formal integration problem* of φ_1 at the point μ_0 . As the system (I) in an infinite system, we try to solve this by induction. For $p \geq 2$ let (I_p) the subsystem given by

$$(I_p) \left\{ \begin{array}{l} \varphi_1 \circ \varphi_1 = -\delta_{\mu_0} \varphi_2 \\ \varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 = -\delta_{\mu_0} \varphi_3 \\ \vdots \\ \sum_{1 \leq i \leq [p/2]} a_{i,p} (\varphi_i \circ \varphi_{p-i} + \varphi_{p-i} \circ \varphi_i) = -\delta_{\mu_0} \varphi_p \end{array} \right.$$

where $a_{i,p} = 1$ if $i \neq p/2$ and $a_{i,p} = 1/2$ if $i = [p/2]$ (this supposes that p is even).

Definition 12 We say that $\varphi_1 \in Z^2(\mu_0, \mu_0)$ is integrable up the order p if there exists $\varphi_i \in C^2(\mu_0, \mu_0)$, $i = 2, \dots, p$ such that (I_p) is satisfied.

Suppose that φ_1 is integrable up the order p . Then we prove directly that

$$\sum_{1 \leq i \leq [p+1/2]} a_{i,p+1}(\varphi_i \circ \varphi_{p+1-i} + \varphi_{p+1-i} \circ \varphi_i) \in Z^3(\mu_0, \mu_0).$$

But φ_1 is integrable up the order $p+1$ if and only if this 3-cochain is in $B^3(\mu_0, \mu_0)$. We deduce the following result:

Proposition 9 If $H^3(\mu_0, \mu_0) = 0$ then every $\varphi_1 \in Z^2(\mu_0, \mu_0)$ is an infinitesimal part of a formal deformation of μ_0 .

The cohomology class $[\sum_{1 \leq i \leq [p+1/2]} a_{i,p+1}(\varphi_i \circ \varphi_{p+1-i} + \varphi_{p+1-i} \circ \varphi_i)]$ is called the obstruction of index $p+1$. The obstruction of index 2 are given by the cohomology class of $\varphi_1 \circ \varphi_1$. It can be written using the following quadratic map:

Definition 13 The Rim quadratic map

$$sq : H^2(\mu_0, \mu_0) \longrightarrow H^3(\mu_0, \mu_0)$$

is defined by

$$sq([\varphi_1]) = [\varphi_1 \circ \varphi_1]$$

for every $\varphi_1 \in Z^2(\mu_0, \mu_0)$.

Using this map, the second obstruction is written $sq([\varphi_1]) = 0$.

Remark. Generalising the notion of formal deformation, we will see in the next section that the infinite system (I) is equivalent to a finite system, that is there exists only a finite number of obstructions.

6.3 Formal equivalence of formal deformations

Let us consider two formal deformations μ_t^1 and μ_t^2 of a law μ_0 . They are called equivalent if there exists a formal linear isomorphism Φ_t of \mathbb{C}^n of the following form

$$\Phi_t = Id + \sum_{i \geq 1} t^i g_i$$

with $g_i \in gl(n, \mathbb{C})$ such that

$$\mu_t^2(X, Y) = \Phi_t^{-1}(\mu_t^1(\Phi_t(X), \Phi_t(Y)))$$

for all $X, Y \in \mathbb{C}^n$.

Definition 14 A deformation μ_t of μ_0 is called trivial if it is equivalent to μ_0 .

In this definition the law μ_0 is considered as a trivial deformation that is when all the bilinear maps φ_i are null.

Let $\mu_t^1 = \mu_0 + \sum_{i=1}^{\infty} t^i \varphi_i$ and $\mu_t^2 = \mu_0 + \sum_{i=1}^{\infty} t^i \psi_i$ be two equivalent deformation of μ_0 . It is easy to see that

$$\varphi_1 - \psi_1 \in B^2(\mu_0, \mu_0).$$

Thus we can consider that the set of infinitesimal parts of deformations is parametrized by $H^2(\mu_0, \mu_0)$.

Suppose now that μ_t is a formal deformation of μ_0 for which $\varphi_t = 0$ for $t = 1, \dots, p$. Then $\delta_{\mu_0} \varphi_{p+1} = 0$. If further $\varphi_{p+1} \in B^2(\mu_0, \mu_0)$, there exists $g \in gl(n, \mathbb{C})$ such that $\delta_{\mu_0} g = \varphi_{p+1}$. Consider the formal isomorphism $\Phi_t = Id + tg$. Then

$$\Phi_t^{-1} \mu_t(\Phi_t, \Phi_t) = \mu_0 + t^{p+2} \varphi_{t+2} + \dots$$

and again $\varphi_{t+2} \in Z^2(\mu_0, \mu_0)$.

Theorem 4 If $Z^2(\mu_0, \mu_0) = 0$, then μ_0 is formally rigid, that is every formal deformation is formally equivalent to μ_0 .

Remarks

1. In the next chapter we will introduce a notion of topological rigidity, that is the orbit of μ_0 is Zariski open. The relation between formal deformations and "topological" deformations is done in by the Nijnhuis Richardson theorem. Before to present this theorem, we will begin to present another notions of deformations, as the perturbations, which are more close of our topological considerations. The link between Gerstenhaber deformations and perturbations is made in the large context of valued deformations (next section).

2. The problem related with the use of formal deformation is the nature of such tool. We can consider a deformation μ_t of a given law μ_0 as a $\mathbb{C}[[t]]$ -algebra on $\mathbb{C}[[t]] \otimes_{\mathbb{C}} \mathbb{C}^n$ given by

$$\mu_t : \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{C}^n \rightarrow \mathbb{C}[[t]] \otimes_{\mathbb{C}} \mathbb{C}^n$$

where

$$\mu_t(X \otimes Y) = \mu_0(X, Y) + t \otimes \varphi_1(X \otimes Y) + t^2 \otimes \varphi_2(X \otimes Y) + \dots$$

But this presentation does not remove the problem concerning the convergence of the formal serie representing μ_t . The first problem we come up against is the resolution of the system (I) defined by an infinity of equations. We will present the deformation differently. Let

$$\mu_t = \mu_0 + \sum_{i=1}^{\infty} t^i \varphi_i$$

a deformation of μ_0 . Amongst the $\{\varphi_i\}_{i \geq 1}$ we extract a free family $\{\varphi_{i_1}, \varphi_{i_2}, \varphi_{i_3}, \dots, \varphi_{i_N}\}$ of $C^2(\mathbb{C}^n, \mathbb{C}^n)$. We can write again the deformation

$$\mu_t = \mu_0 + S_{i_1}(t)\varphi_{i_1} + S_{i_2}(t)\varphi_{i_2} + S_{i_3}(t)\varphi_{i_3} + \dots + S_{i_N}(t)\varphi_{i_N}$$

where $S_{i_j}(t)$ is a formal serie of valuation i_j . Now it is evident that such deformation for which one of the formal series $S_{i_j}(t)$ has a radius of convergence equal to 0 has a limited interest.

Definition 15 A convergent formal deformation μ_t of a Lie algebra law μ_0 is law of L^n defined by

$$\mu_t = \mu_0 + S_1(t)\varphi_1 + S_2(t)\varphi_2 + S_3(t)\varphi_3 + \dots + S_p(t)\varphi_p$$

where

1. $\{\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_p\}$ are linearly independant in $C^2(\mathbb{C}^n, \mathbb{C}^n)$
2. The $S_i(t)$ are power series with a radius of convergence $r_i > 0$
3. The valuation v_i of $S_i(t)$ satisfies $v_i < v_j$ for $i < j$.

Let us put $r = \min\{r_i\}$. For all t in the disc $D_R = \{t \in \mathbb{C}, |t| \leq R < r\}$ the serie μ_t is uniformly convergent. With this viewpoint the map

$$t \in D_R \rightarrow \mu_t \in L^n$$

appears as an analytic curve in L^n passing through μ_0 .

Let us write the Jacobi conditions concerning μ_t . We obtain

$$\mu_t \circ \mu_t = 0 = S_1(t)\delta_{\mu_0}\varphi_1 + [S_1(t)^2]\varphi_1 \circ \varphi_1 + S_2(t)\delta_{\mu_0}\varphi_2 + \dots$$

This identity is true for all $t \in D_R$. By hypothesis, the function

$$f_{ij}(t) = \frac{S_j(t)}{S_i(t)t^{j-i}}$$

is defined by a power serie as soon as $j > i$. This function satisfies $f_{ij}(0) = 0$. Then

Proposition 10 The first term φ_1 of a convergent formal deformation of μ_0 satisfies $\delta_{\mu_0}\varphi_1 = 0$.

Definition 16 The length of the convergent formal deformation

$$\mu_t = \mu_0 + S_1(t)\varphi_1 + S_2(t)\varphi_2 + S_3(t)\varphi_3 + \dots + S_p(t)\varphi_p$$

of μ_0 is the integer p .

For example a convergent formal deformation of length 1 is given by

$$\mu_t = \mu_0 + S_1(t)\varphi_1.$$

It is more general that a linear deformation. In fact the deformation

$$\tilde{\mu}_t = \mu_0 + \sum_{t=1}^{\infty} t^i \varphi_1$$

where all the cochains φ_i are equal to φ_1 corresponds to the convergent formal deformation of length 1

$$\mu_t = \mu_0 + t \frac{1-t^n}{1-t} \varphi_1$$

with $|t| < 1$.

Let us study now the convergent formal deformation of length 2. Such deformation can be written

$$\mu_t = \mu_0 + S_1(t)\varphi_1 + S_2(t)\varphi_2.$$

The equation of convergent formal deformation

$$\begin{aligned} 0 &= S_1(t)\delta_{\mu_0}\varphi_1 + S_1(t)S_2(t)(\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1) + [S_1(t)^2]\varphi_1 \circ \varphi_1 \\ &\quad + S_2(t)\delta_{\mu_0}\varphi_2 + [S_2(t)^2]\varphi_2 \circ \varphi_2 \end{aligned}$$

gives

$$\delta_{\mu_0}\varphi_1 = 0$$

and

$$\begin{aligned} 0 &= S_2(t)(\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1) + [S_1(t)]\varphi_1 \circ \varphi_1 + \frac{S_2(t)}{S_1(t)}\delta_{\mu_0}\varphi_2 \\ &\quad + \frac{[S_2(t)^2]}{S_1(t)}\varphi_2 \circ \varphi_2 \end{aligned}$$

By hypothesis, each fraction is a sum of power series of radius of convergence equal to r . We can compare the two series $S_1(t)$ and $\frac{S_2(t)}{S_1(t)}$, and this depends of the relation between v_1 and $v_2 - v_1$. Let us note that $\delta_{\mu_0}\varphi_2 = 0$ implies that

$$\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 = \varphi_1 \circ \varphi_1 = \varphi_2 \circ \varphi_2 = 0$$

and

$$\mu_t = \mu_0 + S_1(t)\varphi_1$$

is a convergent formal deformation of length 1. Thus we can suppose that $\delta_{\mu_0}\varphi_2 \neq 0$. In this case the 3-cochains $\varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1$, $\varphi_1 \circ \varphi_1$, $\varphi_2 \circ \varphi_2$ generate a system of rank 1 in $C^3(\mathbb{C}^n, \mathbb{C}^n)$. Then the equation of perturbation implies

$$\begin{cases} \delta_{\mu_0}\varphi_1 = 0 \\ \varphi_1 \circ \varphi_1 = \delta_{\mu_0}\varphi_2 \\ \varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 = \delta_{\mu_0}\varphi_2 \\ \varphi_2 \circ \varphi_2 = \delta_{\mu_0}\varphi_2 \end{cases}$$

Proposition 11 1. Let be $\mu_t = \mu_0 + S_1(t)\varphi_1$ a convergent formal deformation of length 1. Then we have

$$\begin{cases} \delta_{\mu_0}\varphi_1 = 0 \\ \varphi_1 \circ \varphi_1 = 0 \end{cases} .$$

2. If $\mu_t = \mu_0 + S_1(t)\varphi_1 + S_2(t)\varphi_2$ is a perturbation of length 2 then we have

$$\begin{cases} \delta_{\mu_0}\varphi_1 = 0 \\ \varphi_1 \circ \varphi_1 = \delta_{\mu_0}\varphi_2 \\ \varphi_1 \circ \varphi_2 + \varphi_2 \circ \varphi_1 = \delta_{\mu_0}\varphi_2 \\ \varphi_2 \circ \varphi_2 = \delta_{\mu_0}\varphi_2 \end{cases} .$$

6.4 Perturbations

The theory of perturbations is based on the infinitesimal framework. The interest of this approach is to have a direct and natural definition of deformations (which will be called in this context perturbations). We will consider Lie algebras whose structural constants are infinitely close to those of a given Lie algebra. This notion can replace the formal notion of deformation and the convergent formal notion. But it is necessary, for a rigorous description of infinitesimal notions, to work in the context of Non Standard Analysis. Here we have two ways. The Robinson approach and the Nelson approach. They are different but they have the same use. In this section we present the notion of perturbation with the Nelson theory because it is more easy. But, in the next section, we introduce the notion of valued deformation which generalize the formal deformations and the perturbations. But here, the Robinson point of view is better because it is more easy to look the good valuation. Well we begin to recall what is the Nelson Non Standard theory.

6.4.1 Non Standard Theory

We start with the axiomatic set theory, that is the set theory subordinated with the Zermelo Fraenkel axioms (ZF) more the choice axiom (C). Nelson construct a new system of axioms, adding of the previous, three news axioms, noted I,S,T as Idealisation, Sandardisation and Transfert and a predicat, noted st as standard, in the Zermelo-Fraenkel vocabulary. But the new theory (ZFC + IST) is consistant respect with the ZFC theory. This is the best result of Nelson. It permits to do classical mathematic in the IST framework. In the IST theory, the objects construct using only ZFC are called standard. Then the sets \mathbb{N} , \mathbb{R} , \mathbb{C} are standard. If we take an element $x \in \mathbb{N}$ which is not standard then, from the construct of Peano of \mathbb{N} , necessary we have $x > n$, for all n standard, $n \in \mathbb{N}$. Such element will be called infinitely large. We deduce a notion of infinitely large in \mathbb{R} or in \mathbb{C} .

Definition 17 *Un element $x \in \mathbb{R}$ or in \mathbb{C} is called infinitely large if it satisfies*

$$x > a, \forall st(a) \in \mathbb{R} \text{ or } \in \mathbb{C}.$$

It is called infinitesimal if it is zero or if x^{-1} is infinitely large.

In other case, it is called appreciable.

These notions can be extended to \mathbb{R}^n or \mathbb{C}^n for n standard. A vector v of \mathbb{R}^n is infinitesimal if all its composante are infiniesimal. It is acelled infinitely large if one of its componants is (this is equivalent to say that its euclidean norm is infinitely large in \mathbb{R}). In the following we denote by IL for an element infinitely large, il for infinitesimal. So we have the following rules

Proposition 12 $il + IL = IL.$

$$IL + IL = IL.$$

$$il \times il = il.$$

$$il / IL = il.$$

On other hand we can in general say nothing about the following cases : $il \times IL$, $IL + IL$, IL / IL .

As a consequence of axiom (S) and the completness of \mathbb{R} , we have the following important property :

Proposition 13 *Every appreciable real element x is infinitely close to a unique standard real element ${}^o x$, called the shadow of x , that is :*

$$\forall x \in \mathbb{R}, x \text{ appreciable, } \exists! {}^o x \text{ standard such that } x - {}^o x \text{ infinitesimal.}$$

If x is infinitesimal, its shadow is 0.

6.4.2 Perturbations of Lie algebra laws

Suppose that μ_0 is a standard Lie algebra law on \mathbb{C}^n , n being supposed standard.

Definition 18 *A perturbation μ of μ_0 is a Lie algebra law on \mathbb{C}^n such that*

$$\mu(X, Y) = \mu_0(X, Y)$$

$\forall X, Y$ standard $\in \mathbb{C}^n$.

Let us consider a standard basis of \mathbb{C}^n . Respect this basis, and also respect any standard basis, the structural constants of μ and μ_0 are infinitely close. Then there exists ϵ infinitesimal such that

$$(\epsilon^{-1}(\mu - \mu_0))(X, Y)$$

are appreciable for all X, Y standard in \mathbb{C}^n . Putting

$$\phi = {}^o(\mu - \mu_0),$$

this bilinear mapping is skew-symmetric and we can write

$$\mu = \mu_0 + \epsilon\phi + \epsilon\psi$$

where ψ is a bilinear skew-symmetric mapping satisfying

$$\psi(X, Y) = 0, \forall X, Y \text{ standard } \in \mathbb{C}^n.$$

As μ is a Lie algebra law, it satisfies the Jacobi condition $\mu \bullet \mu = 0$. Replacing by the expression of μ , we obtain

$$(\mu_0 + \epsilon\phi + \epsilon\psi) \bullet (\mu_0 + \epsilon\phi + \epsilon\psi) = 0.$$

This implies, as $\mu_0 \bullet \mu_0 = 0$,

$$\phi \bullet \mu_0 + \mu_0 \bullet \phi + \psi \bullet \mu_0 + \mu_0 \bullet \psi + \epsilon\phi \bullet \phi + \epsilon\phi \bullet \psi + \epsilon\psi \bullet \phi + \epsilon\psi \bullet \psi = 0.$$

The first part of this equation represents an appreciable trilinear mapping. As it is nul, also its standard part. We deduce that

$$\phi \bullet \mu_0 + \mu_0 \bullet \phi = 2\delta_{\mu_0}\phi = 0.$$

Proposition 14 *Let μ_0 a standard Lie algebra law on \mathbb{C}^n and let μ a perturbation of μ_0 . Then there exist ϵ infinitesimal in \mathbb{C} and an infinitesimal bilinear skew-symmetric mapping ψ such that*

$$\mu = \mu_0 + \epsilon\phi + \epsilon\psi$$

where ϕ is a standard bilinear skew-symmetric mapping satisfying

$$\delta_{\mu_0}\phi = 0.$$

6.5 Valued deformations

6.5.1 Rings of valuation

We recall briefly the classical notion of ring of valuation. Let \mathbb{F} be a (commutative) field and A a subring of \mathbb{F} . We say that A is a ring of valuation of \mathbb{F} if A is a local integral domain satisfying:

$$\text{If } x \in \mathbb{F} - A, \text{ then } x^{-1} \in \mathfrak{m}.$$

where \mathfrak{m} is the maximal ideal of A .

A ring A is called ring of valuation if it is a ring of valuation of its field of fractions.

Examples : Let \mathbb{K} be a commutative field of characteristic 0. The ring of formal series $\mathbb{K}[[t]]$ is a valuation ring. On other hand the ring $\mathbb{K}[[t_1, t_2]]$ of two (or more) indeterminates is not a valuation ring.

6.5.2 Valued deformations of Lie algebra

Let \mathfrak{g} be a \mathbb{K} -Lie algebra and A a commutative \mathbb{K} -algebra of valuation. Then $\mathfrak{g} \otimes A$ is a \mathbb{K} -Lie algebra. We can consider this Lie algebra as an A -Lie algebra. We denote this last by \mathfrak{g}_A . If $\dim_{\mathbb{K}}(\mathfrak{g})$ is finite then

$$\dim_A(\mathfrak{g}_A) = \dim_{\mathbb{K}}(\mathfrak{g}).$$

As the valued ring A is also a \mathbb{K} -algebra we have a natural embedding of the \mathbb{K} -vector space \mathfrak{g} into the free A -module \mathfrak{g}_A . Without loss of generality we can consider this embedding to be the identity map.

Definition 19 *Let \mathfrak{g} be a \mathbb{K} -Lie algebra and A a commutative \mathbb{K} -algebra of valuation such that the residual field $\frac{A}{\mathfrak{m}}$ is isomorphic to \mathbb{K} (or to a subfield of \mathbb{K}). A valued deformation of \mathfrak{g} with base A is a A -Lie algebra \mathfrak{g}'_A such that the underlying A -module of \mathfrak{g}'_A is \mathfrak{g}_A and that*

$$[X, Y]_{\mathfrak{g}'_A} - [X, Y]_{\mathfrak{g}_A}$$

is in the \mathfrak{m} -quasi-module $\mathfrak{g} \otimes \mathfrak{m}$ where \mathfrak{m} is the maximal ideal of A .

Examples

1. Formal deformations. The classical notion of deformation studied by Gerstenhaber is a valued deformation. In this case $A = \mathbb{K}[[t]]$ and the residual field of A is isomorphic to \mathbb{K} . Likewise a versal deformation is a valued deformation. The algebra A is in this case the finite dimensional \mathbb{K} -vector space

$\mathbb{K} \oplus (H^2(\mathfrak{g}, \mathfrak{g}))^*$ where H^2 denotes the second Chevalley cohomology group of \mathfrak{g} . The algebra law is given by

$$(\alpha_1, h_1).(\alpha_2, h_2) = (\alpha_1.\alpha_2, \alpha_1.h_2 + \alpha_2.h_1).$$

It is a local field with maximal ideal $\{0\} \oplus (H^2)^*$. It is also a valuation field because we can endow this algebra with a field structure, the inverse of (α, h) being $((\alpha)^{-1}, -(\alpha)^{-2}h)$.

2. Versal deformations of Fialowski[?]. Let \mathfrak{g} be a \mathbb{K} -Lie algebra and A an unitary commutative local \mathbb{K} -algebra. The tensor product $\mathfrak{g} \otimes A$ is naturally endowed with a Lie algebra structure :

$$[X \otimes a, Y \otimes b] = [X, Y] \otimes ab.$$

If $\epsilon : A \rightarrow \mathbb{K}$, is an unitary augmentation with kernel the maximal ideal \mathfrak{m} , a deformation λ of \mathfrak{g} with base A is a Lie algebra structure on $\mathfrak{g} \otimes A$ with bracket $[\cdot, \cdot]_\lambda$ such that

$$id \otimes \epsilon : \mathfrak{g} \otimes A \rightarrow \mathfrak{g} \otimes \mathbb{K}$$

is a Lie algebra homomorphism. In this case the bracket $[\cdot, \cdot]_\lambda$ satisfies

$$[X \otimes 1, Y \otimes 1]_\lambda = [X, Y] \otimes 1 + \sum Z_i \otimes a_i$$

where $a_i \in A$ and $X, Y, Z_i \in \mathfrak{g}$. Such a deformation is called infinitesimal if the maximal ideal \mathfrak{m} satisfies $\mathfrak{m}^2 = 0$. An interesting example is described in [F]. If we consider the commutative algebra $A = \mathbb{K} \oplus (H^2(\mathfrak{g}, \mathfrak{g}))^*$ (where $*$ denotes the dual as vector space) such that $dim(H^2) \leq \infty$, the deformation with base A is an infinitesimal deformation (which plays the role of an universal deformation).

3. Perturbations. Let \mathbb{C}^* be a non standard extension of \mathbb{C} in the Robinson sense [?]. If \mathbb{C}_l is the subring of non-infinitely large elements of \mathbb{C}^* then the subring \mathfrak{m} of infinitesimals is the maximal ideal of \mathbb{C}_l and \mathbb{C}_l is a valued ring. Let us consider $A = \mathbb{C}_l$. In this case we have a natural embedding of the variety of A -Lie algebras in the variety of \mathbb{C} -Lie algebras. Up this embedding (called the transfert principle in the Robinson theory), the set of A -deformations of \mathfrak{g}_A is an infinitesimal neighbourhood of \mathfrak{g} contained in the orbit of \mathfrak{g} . Thus any perturbation can be appear as a valued deformation.

6.6 Decomposition of valued deformations

In this section we show that every valued deformation can be decomposed in a finite sum (and not as a serie) with pairwise comparable infinitesimal coefficients (that is in \mathfrak{m}). The interest of this decomposition is to avoid the classical problems of convergence.

6.6.1 Decomposition in $\mathfrak{m} \times \mathfrak{m}$

Let A be a valuation ring satisfying the conditions of definition 1. Let us denote by \mathcal{F}_A the field of fractions of A and \mathfrak{m}^2 the catesian product $\mathfrak{m} \times \mathfrak{m}$. Let $(a_1, a_2) \in \mathfrak{m}^2$ with $a_i \neq 0$ for $i = 1, 2$.

i) Suppose that $a_1.a_2^{-1} \in A$. Let be $\alpha = \pi(a_1.a_2^{-1})$ where π is the canonical projection on $\frac{A}{\mathfrak{m}}$. Clearly, there exists a global section $s : \mathbb{K} \rightarrow A$ which permits to identify α with $s(\alpha)$ in A . Then

$$a_1.a_2^{-1} = \alpha + a_3$$

with $a_3 \in \mathfrak{m}$. Then if $a_3 \neq 0$,

$$(a_1, a_2) = (a_2(\alpha + a_3), a_2) = a_2(\alpha, 1) + a_2a_3(0, 1).$$

If $\alpha \neq 0$ we can also write

$$(a_1, a_2) = aV_1 + abV_2$$

with $a, b \in \mathfrak{m}$ and V_1, V_2 linearly independent in \mathbb{K}^2 . If $\alpha = 0$ then $a_1.a_2^{-1} \in \mathfrak{m}$ and $a_1 = a_2a_3$. We have

$$(a_1, a_2) = (a_2a_3, a_2) = ab(1, 0) + a(0, 1).$$

So in this case, $V_1 = (0, 1)$ and $V_2 = (1, 0)$. If $a_3 = 0$ then

$$a_1a_2^{-1} = \alpha$$

and

$$(a_1, a_2) = a_2(\alpha, 1) = aV_1.$$

This correspond to the previous decomposition but with $b = 0$.

ii) If $a_1.a_2^{-1} \in \mathcal{F}_A - A$, then $a_2.a_1^{-1} \in \mathfrak{m}$. We put in this case $a_2.a_1^{-1} = a_3$ and we have

$$(a_1, a_2) = (a_1, a_1.a_3) = a_1(1, a_3) = a_1(1, 0) + a_1a_3(0, 1)$$

with $a_3 \in \mathfrak{m}$. Then, in this case the point (a_1, a_2) admits the following decomposition :

$$(a_1, a_2) = aV_1 + abV_2$$

with $a, b \in \mathfrak{m}$ and V_1, V_2 linearly independent in \mathbb{K}^2 . Note that this case corresponds to the previous but with $\alpha = 0$.

Then we have proved

Proposition 15 *For every point $(a_1, a_2) \in \mathfrak{m}^2$, there exist linearly independent vectors V_1 and V_2 in the \mathbb{K} -vector space \mathbb{K}^2 such that*

$$(a_1, a_2) = aV_1 + abV_2$$

for some $a, b \in \mathfrak{m}$.

Such decomposition est called of length 2 if $b \neq 0$. If not it is called of length 1.

6.6.2 Decomposition in \mathfrak{m}^k

Suppose that A is valuation ring satisfying the hypothesis of Definition 1. Arguing as before, we can conclude

Theorem 5 *For every $(a_1, a_2, \dots, a_k) \in \mathfrak{m}^k$ there exist h ($h \leq k$) independent vectors V_1, V_2, \dots, V_h whose components are in \mathbb{K} and elements $b_1, b_2, \dots, b_h \in \mathfrak{m}$ such that*

$$(a_1, a_2, \dots, a_k) = b_1V_1 + b_1b_2V_2 + \dots + b_1b_2\dots b_hV_h.$$

The parameter h which appears in this theorem is called the length of the decomposition. This parameter can be different to k . It corresponds to the dimension of the smallest \mathbb{K} -vector space V such that $(a_1, a_2, \dots, a_k) \in V \otimes \mathfrak{m}$.

If the coordinates a_i of the vector (a_1, a_2, \dots, a_k) are in A and not necessarily in its maximal ideal, then writing $a_i = \alpha_i + a'_i$ with $\alpha_i \in \mathbb{K}$ and $a'_i \in \mathfrak{m}$, we decompose

$$(a_1, a_2, \dots, a_k) = (\alpha_1, \alpha_2, \dots, \alpha_k) + (a'_1, a'_2, \dots, a'_k)$$

and we can apply Theorem 1 to the vector $(a'_1, a'_2, \dots, a'_k)$.

6.6.3 Uniqueness of the decomposition

Let us begin by a technical lemma.

Lemma 1 *Let V and W be two vectors with components in the valuation ring A . There exist V_0 and W_0 with components in \mathbb{K} such that $V = V_0 + V'_0$ and $W = W_0 + W'_0$ and the components of V'_0 and W'_0 are in the maximal ideal \mathfrak{m} . Moreover if the vectors V_0 and W_0 are linearly independent then V and W are also independent.*

Proof. The decomposition of the two vectors V and W is evident. It remains to prove that the independence of the vectors V_0 and W_0 implies those of V and W . Let V, W be two vectors with components in A such that $\pi(V) = V_0$ and $\pi(W) = W_0$ are independent. Let us suppose that

$$xV + yW = 0$$

with $x, y \in A$. One of the coefficients xy^{-1} or yx^{-1} is not in \mathfrak{m} . Let us suppose that $xy^{-1} \notin \mathfrak{m}$. If $xy^{-1} \notin A$ then $x^{-1}y \in \mathfrak{m}$. Then $xV + yW = 0$ is equivalent to $V + x^{-1}yW = 0$. This implies that $\pi(V) = 0$ and this is impossible. Then $xy^{-1} \in A - \mathfrak{m}$. Thus if there exists a linear relation between V and W , there exists a linear relation with coefficients in $A - \mathfrak{m}$. We can suppose that $xV + yW = 0$ with $x, y \in A - \mathfrak{m}$. As $V = V_0 + V'_0$, $W = W_0 + W'_0$ we have

$$\pi(xV + yW) = \pi(x)V_0 + \pi(y)W_0 = 0.$$

Thus $\pi(x) = \pi(y) = 0$. This is impossible and the vectors V and W are independent as soon as V_0 and W_0 are independent vectors. \square

Let $(a_1, a_2, \dots, a_k) = b_1V_1 + b_2V_2 + \dots + b_hV_h$ and $(a_1, a_2, \dots, a_k) = c_1W_1 + c_2W_2 + \dots + c_sW_s$ be two decompositions of the vector (a_1, a_2, \dots, a_k) . Let us compare the coefficients b_1 and c_1 . By hypothesis $b_1c_1^{-1}$ is in A or the inverse is in \mathfrak{m} . Then we can suppose that $b_1c_1^{-1} \in A$. As the residual field is a subfield of \mathbb{K} , there exists $\alpha \in \frac{A}{\mathfrak{m}}$ and $c_1 \in \mathfrak{m}$ such that

$$b_1c_1^{-1} = \alpha + b_{11}$$

thus $b_1 = \alpha c_1 + b_{11}c_1$. Replacing this term in the decompositions we obtain

$$\begin{aligned} & (\alpha c_1 + b_{11}c_1)V_1 + (\alpha c_1 + b_{11}c_1)b_2V_2 + \dots + (\alpha c_1 + b_{11}c_1)b_hV_h \\ & = c_1W_1 + c_2W_2 + \dots + c_sW_s. \end{aligned}$$

Simplifying by c_1 , this expression is written

$$\alpha V_1 + m_1 = W_1 + m_2$$

where m_1, m_2 are vectors with coefficients $\in \mathfrak{m}$. From Lemma 1, if V_1 and W_1 are linearly independent, as its coefficients are in the residual field, the vectors $\alpha V_1 + m_1$ and $W_1 + m_2$ would be also linearly independent ($\alpha \neq 0$). Thus $W_1 = \alpha V_1$. One deduces

$$b_1 V_1 + b_1 b_2 V_2 + \dots + b_1 b_2 \dots b_h V_h = c_1(\alpha V_1) + c_1 b_{11} V_1 + c_1 b_{12} V_2 + \dots + c_1 b_{12} b_3 \dots b_h V_h,$$

with $b_{12} = b_2(\alpha + b_{11})$. Then

$$b_{11} V_1 + b_{12} V_2 + \dots + b_{12} b_3 \dots b_h V_h = c_2 W_2 + \dots + c_2 \dots c_s W_s.$$

Continuing this process by induction we deduce the following result

Theorem 6 *Let be $b_1 V_1 + b_1 b_2 V_2 + \dots + b_1 b_2 \dots b_h V_h$ and $c_1 W_1 + c_1 c_2 W_2 + \dots + c_1 c_2 \dots c_s W_s$ two decompositions of the vector (a_1, a_2, \dots, a_k) . Then*

- i. $h = s$,
- ii. *The flag generated by the ordered free family (V_1, V_2, \dots, V_h) is equal to the flag generated by the ordered free family (W_1, W_2, \dots, W_h) that is $\forall i \in 1, \dots, h$*

$$\{V_1, \dots, V_i\} = \{W_1, \dots, W_i\}$$

where $\{U_i\}$ designates the linear space generated by the vectors U_i .

6.6.4 Geometrical interpretation of this decomposition

Let A be an \mathbb{R} algebra of valuation. Consider a differential curve γ in \mathbb{R}^3 . We can embed γ in a differential curve

$$\Gamma : \mathbb{R} \otimes A \rightarrow \mathbb{R}^3 \otimes A.$$

Let $t = t_0 \otimes 1 + 1 \otimes \epsilon$ an parameter infinitely close to t_0 , that is $\epsilon \in \mathfrak{m}$. If M corresponds to the point of Γ of parameter t and M_0 those of t_0 , then the coordinates of the point $M - M_0$ in the affine space $\mathbb{R}^3 \otimes A$ are in $\mathbb{R} \otimes \mathfrak{m}$. In the flag associated to the decomposition of $M - M_0$ we can consider a direct orthonormal frame (V_1, V_2, V_3) . It is the Serret-Frenet frame to γ at the point M_0 .

6.6.5 Decomposition of a valued deformation of a Lie algebra

Let \mathfrak{g}'_A be a valued deformation with base A of the \mathbb{K} -Lie algebra \mathfrak{g} . By definition, for every X and Y in \mathfrak{g} we have $[X, Y]_{\mathfrak{g}'_A} - [X, Y]_{\mathfrak{g}_A} \in \mathfrak{g} \otimes \mathfrak{m}$. Suppose that \mathfrak{g} is finite dimensional and let $\{X_1, \dots, X_n\}$ be a basis of \mathfrak{g} . In this case

$$[X_i, X_j]_{\mathfrak{g}'_A} - [X_i, X_j]_{\mathfrak{g}_A} = \sum_k C_{ij}^k X_k$$

with $C_{ij}^k \in \mathfrak{m}$. Using the decomposition of the vector of $\mathfrak{m}^{n^2(n-1)/2}$ with for components C_{ij}^k , we deduce that

$$\begin{aligned} [X_i, X_j]_{\mathfrak{g}'_A} - [X_i, X_j]_{\mathfrak{g}_A} &= a \epsilon_1 \phi_1(X_i, X_j) + \epsilon_1 \epsilon_2 \phi_2(X_i, X_j) \\ &+ \dots + \epsilon_1 \epsilon_2 \dots \epsilon_k \phi_k(X_i, X_j) \end{aligned}$$

where $\epsilon_s \in \mathfrak{m}$ and ϕ_1, \dots, ϕ_l are linearly independent. Then we have

$$\begin{aligned} [X, Y]_{\mathfrak{g}'_A} - [X, Y]_{\mathfrak{g}_A} &= \epsilon_1 \phi_1(X, Y) + \epsilon_1 \epsilon_2 \phi_2(X, Y) \\ &+ \dots + \epsilon_1 \epsilon_2 \dots \epsilon_k \phi_k(X, Y) \end{aligned}$$

where the bilinear maps ϵ_i have values in \mathfrak{m} and linear maps $\phi_i : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ are linearly independent.

If \mathfrak{g} is infinite dimensional with a countable basis $\{X_n\}_{n \in \mathbb{N}}$ then the \mathbb{K} -vector space of linear map $T_2^1 = \{\phi : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}\}$ also admits a countable basis.

Theorem 7 *If $\mu_{\mathfrak{g}'_A}$ (resp. $\mu_{\mathfrak{g}_A}$) is the law of the Lie algebra \mathfrak{g}'_A (resp. \mathfrak{g}_A) then*

$$\mu_{\mathfrak{g}'_A} - \mu_{\mathfrak{g}_A} = \sum_{i \in I} \epsilon_1 \epsilon_2 \dots \epsilon_i \phi_i$$

where I is a finite set of indices, $\epsilon_i : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{m}$ are linear maps and ϕ_i 's are linearly independent maps in T_2^1 .

6.6.6 Equations of valued deformations

We will prove that the classical equations of deformation given by Gerstenhaber are still valid in the general frame of valued deformations. Nevertheless we can prove that the infinite system described by Gerstenhaber and which gives the conditions to obtain a deformation, can be reduced to a system of finite rank. Let

$$\mu_{\mathfrak{g}'_A} - \mu_{\mathfrak{g}_A} = \sum_{i \in I} \epsilon_1 \epsilon_2 \dots \epsilon_i \phi_i$$

be a valued deformation of μ (the bracket of \mathfrak{g}). Then $\mu_{\mathfrak{g}'_A}$ satisfies the Jacobi equations. Following Gerstenhaber we consider the Chevalley-Eilenberg graded differential complex $\mathcal{C}(\mathfrak{g}, \mathfrak{g})$ and the product \circ defined by

$$(g_q \circ f_p)(X_1, \dots, X_{p+q}) = \sum (-1)^{\epsilon(\sigma)} g_q(f_p(X_{\sigma(1)}, \dots, X_{\sigma(p)}), X_{\sigma(p+1)}, \dots, X_{\sigma(q)})$$

where σ is a permutation of $1, \dots, p+q$ such that $\sigma(1) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(p+q)$ (it is a (p, q) -schuffle); $g_q \in \mathcal{C}^q(\mathfrak{g}, \mathfrak{g})$ and $f_p \in \mathcal{C}^p(\mathfrak{g}, \mathfrak{g})$. As $\mu_{\mathfrak{g}'_A}$ satisfies the Jacobi identities, $\mu_{\mathfrak{g}'_A} \circ \mu_{\mathfrak{g}'_A} = 0$. This gives

$$(\mu_{\mathfrak{g}_A} + \sum_{i \in I} \epsilon_1 \epsilon_2 \dots \epsilon_i \phi_i) \circ (\mu_{\mathfrak{g}_A} + \sum_{i \in I} \epsilon_1 \epsilon_2 \dots \epsilon_i \phi_i) = 0. \quad (1)$$

As $\mu_{\mathfrak{g}_A} \circ \mu_{\mathfrak{g}_A} = 0$, this equation becomes :

$$\epsilon_1 (\mu_{\mathfrak{g}_A} \circ \phi_1 + \phi_1 \circ \mu_{\mathfrak{g}_A}) + \epsilon_1 U = 0$$

where U is in $\mathcal{C}^3(\mathfrak{g}, \mathfrak{g}) \otimes \mathfrak{m}$. If we symplify by ϵ_1 which is supposed non zero if not the deformation is trivial, we obtain

$$(\mu_{\mathfrak{g}_A} \circ \phi_1 + \phi_1 \circ \mu_{\mathfrak{g}_A})(X, Y, Z) + U(X, Y, Z) = 0$$

for all $X, Y, Z \in \mathfrak{g}$. As $U(X, Y, Z)$ is in the module $\mathfrak{g} \otimes \mathfrak{m}$ and the first part in $\mathfrak{g} \otimes A$, each one of these vectors is null. Then

$$(\mu_{\mathfrak{g}_A} \circ \phi_1 + \phi_1 \circ \mu_{\mathfrak{g}_A})(X, Y, Z) = 0.$$

Proposition 16 *For every valued deformation with base A of the \mathbb{K} -Lie algebra \mathfrak{g} , the first term ϕ appearing in the associated decomposition is a 2-cochain of the Chevalley-Eilenberg cohomology of \mathfrak{g} belonging to $Z^2(\mathfrak{g}, \mathfrak{g})$.*

We thus rediscover the classical result of Gerstenhaber but in the broader context of valued deformations and not only for the valued deformation of basis the ring of formal series.

In order to describe the properties of other terms of equations (1) we use the super-bracket of Gerstenhaber which endows the space of Chevalley-Eilenberg cochains $\mathcal{C}(\mathfrak{g}, \mathfrak{g})$ with a Lie superalgebra structure. When $\phi_i \in \mathcal{C}^2(\mathfrak{g}, \mathfrak{g})$, it is defines by

$$[\phi_i, \phi_j] = \phi_i \circ \phi_j + \phi_j \circ \phi_i$$

and $[\phi_i, \phi_j] \in \mathcal{C}^3(\mathfrak{g}, \mathfrak{g})$.

Lemma 2 *Let us suppose that $I = \{1, \dots, k\}$. If*

$$\mu_{\mathfrak{g}'_A} = \mu_{\mathfrak{g}_A} + \sum_{i \in I} \epsilon_1 \epsilon_2 \dots \epsilon_i \phi_i$$

is a valued deformation of μ , then the 3-cochains $[\phi_i, \phi_j]$ and $[\mu, \phi_i]$, $1 \leq i, j \leq k-1$, generate a linear subspace V of $\mathcal{C}^3(\mathfrak{g}, \mathfrak{g})$ of dimension less or equal to $k(k-1)/2$. Moreover, the 3-cochains $[\phi_i, \phi_j]$, $1 \leq i, j \leq k-1$, form a system of generators of this space.

Proof. Let V be the subspace of $\mathcal{C}^3(\mathfrak{g}, \mathfrak{g})$ generated by $[\phi_i, \phi_j]$ and $[\mu, \phi_i]$. If ω is a linear form on V of which kernel contains the vectors $[\phi_i, \phi_j]$ for $1 \leq i, j \leq (k-1)$, then the equation (1) gives

$$\begin{aligned} \epsilon_1 \epsilon_2 \dots \epsilon_k \omega([\phi_1, \phi_k]) + \epsilon_1 \epsilon_2^2 \dots \epsilon_k \omega([\phi_2, \phi_k]) + \dots + \epsilon_1 \epsilon_2^2 \dots \epsilon_k^2 \omega([\phi_k, \phi_k]) + \epsilon_2 \omega([\mu, \phi_2]) \\ + \epsilon_2 \epsilon_3 \omega([\mu, \phi_3]) \dots + \epsilon_2 \epsilon_3 \dots \epsilon_k \omega([\mu, \phi_k]) = 0. \end{aligned}$$

As the coefficients which appear in this equation are each one in one \mathfrak{m}^p , we have necessarily

$$\omega([\phi_1, \phi_k]) = \dots = \omega([\phi_k, \phi_k]) = \omega([\mu, \phi_2]) = \dots = \omega([\mu, \phi_k]) = 0$$

and this for every linear form ω of which kernel contains V . This proves the lemma.

From this lemma and using the descending sequence

$$\mathfrak{m} \supset \mathfrak{m}^{(2)} \supset \dots \supset \mathfrak{m}^{(p)} \dots$$

where $\mathfrak{m}^{(p)}$ is the ideal generated by the products $a_1 a_2 \dots a_p$, $a_i \in \mathfrak{m}$ of length p , we obtain :

Proposition 17 *If*

$$\mu_{\mathfrak{g}'_A} = \mu_{\mathfrak{g}_A} + \sum_{i \in I} \epsilon_1 \epsilon_2 \dots \epsilon_i \phi_i$$

is a valued deformation of μ , then we have the following linear system :

$$\begin{cases} \delta\phi_1 = 0 \\ \delta\phi_2 = a_{11}^2[\phi_1, \phi_1] \\ \delta\phi_3 = a_{12}^3[\phi_1, \phi_2] + a_{22}^3[\phi_1, \phi_1] \\ \dots \\ \delta\phi_k = \sum_{1 \leq i \leq j \leq k-1} a_{ij}^k[\phi_i, \phi_j] \\ [\phi_1, \phi_k] = \sum_{1 \leq i \leq j \leq k-1} b_{ij}^1[\phi_i, \phi_j] \\ \dots \\ [\phi_{k-1}, \phi_k] = \sum_{1 \leq i \leq j \leq k-1} b_{ij}^{k-1}[\phi_i, \phi_j] \end{cases}$$

where $\delta\phi_i = [\mu, \phi_i]$ is the coboundary operator of the Chevalley cohomology of the Lie algebra \mathfrak{g} .

Let us suppose that the dimension of V is the maximum $k(k-1)/2$. In this case we have no other relations between the generators of V and the previous linear system is complete, that is the equation of deformations does not give other relations than the relations of this system. The following result shows that, in this case, such deformation is isomorphic, as Lie algebra laws, to a "polynomial" valued deformation.

Proposition 18 *Let be $\mu_{\mathfrak{g}'_A}$ a valued deformation of μ such that*

$$\mu_{\mathfrak{g}'_A} = \mu_{\mathfrak{g}_A} + \sum_{i=1, \dots, k} \epsilon_1 \epsilon_2 \dots \epsilon_i \phi_i$$

and $\dim V = k(k-1)/2$. Then there exists an automorphism of $\mathbb{K}^n \otimes \mathfrak{m}$ of the form $f = Id \otimes P_k(\epsilon)$ with $P_k(X) \in \mathbb{K}^k[X]$ satisfying $P_k(0) = 1$ and $\epsilon \in \mathfrak{m}$ such that the valued deformation $\mu_{\mathfrak{g}'_A}$ defined by

$$\mu_{\mathfrak{g}''_A}(X, Y) = f^{-1}(\mu_{\mathfrak{g}'_A}(f(X), f(Y)))$$

is of the form

$$\mu_{\mathfrak{g}''_A} = \mu_{\mathfrak{g}_A} + \sum_{i=1, \dots, k} \epsilon^i \varphi_i$$

where $\varphi_i = \sum_{j \leq i} \phi_j$.

Proof. Considering the Jacobi equation

$$[\mu_{\mathfrak{g}'_A}, \mu_{\mathfrak{g}'_A}] = 0$$

and writting that $\dim V = k(k-1)/2$, we deduce that there exist polynomials $P_i(X) \in \mathbb{K}[X]$ of degree i such that

$$\epsilon_i = a_i \epsilon_k \frac{P_{k-i}(\epsilon_k)}{P_{k-i+1}(\epsilon_k)}$$

with $a_i \in \mathbb{K}$. Then we have

$$\mu_{\mathfrak{g}'_A} = \mu_{\mathfrak{g}_A} + \sum_{i=1, \dots, k} a_1 a_2 \dots a_i (\epsilon_k)^i \frac{P_{k-i}(\epsilon_k)}{P_k(\epsilon_k)} \phi_i.$$

Thus

$$P_k(\epsilon_k) \mu_{\mathfrak{g}'_A} = P_k(\epsilon_k) \mu_{\mathfrak{g}_A} + \sum_{i=1, \dots, k} a_1 a_2 \dots a_i (\epsilon_k)^i P_{k-i}(\epsilon_k) \phi_i.$$

If we write this expression according the increasing powers we obtain the announced expression. \square
Let us note that, for such deformation we have

$$\begin{cases} \delta\varphi_2 + [\varphi_1, \varphi_1] = 0 \\ \delta\varphi_3 + [\varphi_1, \varphi_2] = 0 \\ \dots \\ \delta\varphi_k + \sum_{i+j=k} [\varphi_i, \varphi_j] = 0 \\ \sum_{i+j=k+s} [\varphi_i, \varphi_j] = 0. \end{cases}$$

6.6.7 Particular case : one-parameter deformations of Lie algebras

In this section the valuation ring A is $\mathbb{K}[[t]]$. Its maximal ideal is $t\mathbb{K}[[t]]$ and the residual field is \mathbb{K} . Let \mathfrak{g} be a \mathbb{K} - Lie algebra. Consider $\mathfrak{g} \otimes A$ as an A -algebra and let be \mathfrak{g}'_A a valued deformation of \mathfrak{g} . The bracket $[\cdot, \cdot]_t$ of this Lie algebra satisfies

$$[X, Y]_t = [X, Y] + \sum t^i \phi_i(X, Y).$$

Considered as a valued deformation with base $\mathbb{K}[[t]]$, this bracket can be written

$$[X, Y]_t = [X, Y] + \sum_{i=1}^{i=k} c_i(t) \dots c_i(t) \psi_i(X, Y)$$

where (ψ_1, \dots, ψ_k) are linearly independent and $c_i(t) \in t\mathbb{C}[[t]]$. As $\phi_1 = \psi_1$, this bilinear map belongs to $Z^2(\mathfrak{g}, \mathfrak{g})$ and we find again the classical result of Gerstenhaber. Let V be the \mathbb{K} -vector space generated by $[\phi_i, \phi_j]$ and $[\mu, \phi_i]$, $i, j = 1, \dots, k-1$, μ being the law of \mathfrak{g} . If $\dim V = k(k-1)/2$ we will say that one-parameter deformation $[\cdot, \cdot]_t$ is of maximal rank.

Proposition 19 *Let*

$$[X, Y]_t = [X, Y] + \sum t^i \phi_i(X, Y)$$

be a one-parameter deformation of \mathfrak{g} . If its rank is maximal then this deformation is equivalent to a polynomial deformation

$$[X, Y]'_t = [X, Y] + \sum_{i=1, \dots, k} t^i \varphi_i$$

with $\varphi_i = \sum_{j=1, \dots, i} a_{ij} \psi_j$.

Corollary 2 *Every one-parameter deformation of maximal rank is equivalent to a local non valued deformation with base the local algebra $\mathbb{K}[t]$.*

Recall that the algebra $\mathbb{K}[t]$ is not an algebra of valuation. But every local ring is dominated by a valuation ring. Then this corollary can be interpreted as saying that every deformation in the local algebra $\mathbb{C}[t]$ of polynomials with coefficients in \mathbb{C} is equivalent to a "classical"-Gerstenhaber deformation with maximal rank.

7 The scheme \mathcal{L}^n and the cohomology $H(L^n, L^n)$

In this section, we are interested by geometrical properties of \mathcal{L}^n and L^n which can be expressed using the Chevalley cohomology $H(\mathfrak{g}, \mathfrak{g})$. Let x be a point of the scheme \mathcal{L}^n and $\mathfrak{g} \in L^n$ the corresponding Lie algebra. The algebraic group $Gl(n, \mathbb{C})$ operates on L^n and we have noted by $\mathcal{O}(\mathfrak{g})$ the orbit of \mathfrak{g} . The scheme corresponding to the algebraic variety $Gl(n, \mathbb{C})$ is reduced (every group scheme is reduced because the characteristic of the field is 0) and is nothing that as the variety itself. Then, if we denote by $T_x^0(\mathcal{L}^n)$ the tangent space to the orbit scheme of x at the point x , we have :

Proposition 20 [?] *There is a canonical isomorphism*

$$\frac{T_x(\mathcal{L}^n)}{T_x^0(\mathcal{L}^n)} \simeq H^2(\mathfrak{g}, \mathfrak{g}).$$

Let x be in \mathcal{L}^n and \mathfrak{g} the corresponding Lie algebra.

Lemma 3 *If $H^p(\mathfrak{g}, \mathfrak{g}) = 0$, then there exists a neighborhood U of x in \mathcal{L}^n such as $H^p(\mathfrak{g}_y, \mathfrak{g}_y) = 0$ for all Lie algebra \mathfrak{g}_y corresponding to the points $y \in U$.*

Indeed, the function

$$\delta^i : x \in \mathcal{L}^n \rightarrow \dim H^i(\mathfrak{g}, \mathfrak{g})$$

is upper semicontinuous. Then, if $H^p(\mathfrak{g}, \mathfrak{g}) = 0$, this implies that there is a neighborhood of x such that $\dim H^p(\mathfrak{g}_y, \mathfrak{g}_y) = 0$ for every Lie algebra \mathfrak{g}_y associated to the points y of U .

Theorem 8 *Let \mathfrak{g} be in L^n with $H^1(\mathfrak{g}, \mathfrak{g}) = 0$. Then there is an open neighborhood U of x in \mathcal{L}^n such as the dimension of the orbits of the points $y \in U$ (and then of the Lie algebras \mathfrak{g}_y) are constant and equal to $n^2 - n + \dim H^0(\mathfrak{g}, \mathfrak{g})$.*

Indeed, if $H^1(\mathfrak{g}, \mathfrak{g}) = 0$, from the previous lemma, $H^1(\mathfrak{g}_y, \mathfrak{g}_y) = 0$ for every Lie algebra \mathfrak{g}_y associated to the points y of U . Thus we have

$$\dim H^0(\mathfrak{g}_y, \mathfrak{g}_y) = \dim H^0(\mathfrak{g}, \mathfrak{g})$$

As the codimension of the orbit of \mathfrak{g}_y is $n - \dim H^0(\mathfrak{g}_y, \mathfrak{g}_y)$, we have $\text{codim} \mathcal{O}(\mathfrak{g}_y) = \text{codim} \mathcal{O}(\mathfrak{g}) = n - \dim H^0(\mathfrak{g}, \mathfrak{g})$.

Corollary 3 *If \mathfrak{g} satisfies $H^1(\mathfrak{g}, \mathfrak{g}) = 0$ and if \mathfrak{g} is a contraction of \mathfrak{g}' , then \mathfrak{g} and \mathfrak{g}' are isomorphic.*

This follows from the definition of the contraction : $\mathfrak{g} \in \overline{\mathcal{O}(\mathfrak{g}')}$. As $U \cap \mathcal{O}(\mathfrak{g}') \neq \emptyset$, $\dim \mathcal{O}(\mathfrak{g}') = \dim \mathcal{O}(\mathfrak{g})$. Thus $\mathcal{O}(\mathfrak{g}') = \mathcal{O}(\mathfrak{g})$.

8 Rigid Lie algebras

8.1 Definition

Let $\mathfrak{g} = (\mathbb{C}^n, \mu)$ be a n -dimensional complex Lie algebra. We note always by μ the corresponding point of L_n .

Definition 20 *The Lie algebra \mathfrak{g} is called rigid if its orbit $\mathcal{O}(\mu)$ is open (for the Zariski topology) in L_n .*

Example. Let us consider the 2-dimensional Lie algebra given by

$$\mu(e_1, e_2) = e_2.$$

We have seen in the previous section that its orbit is the set of Lie algebra products given by

$$\mu_{a,b}(e_1, e_2) = ae_1 + be_2$$

for all $a, b \in \mathbb{C}$. This orbit is open then μ is rigid.

Remark. It is no easy to work with the Zariski topology (the closed set are given by polynomial equations). But, in our case, we can bypass the problem. Indeed, the orbit of a point is an homogeneous space and thus provided with a differential structure. In this case the topology of Zariski and the metric topology induced by the metric topology of the vector space \mathbb{C}^N of structure constants coincide. Then μ is rigid if $\mathcal{O}(\mu)$ is open in \mathbb{C}^N .

8.2 Rigidity and Deformations

Intuitively a Lie algebra is rigid if any close Lie algebra is the same or isomorphic. We have presented the notion of deformations as a notion which try to describe what is a close Lie algebra. Then we shall have criterium of rigidity in terms of deformations or perturbations. We recall that a perturbation is a special valued deformation whose valued ring of coefficients is given by the ring of limited elements in a non standard extension of the complex nuber.

Theorem 9 *Let $\mathfrak{g} = (\mathbb{C}^n, \mu)$ a n -dimensional complex Lie algebra. Then \mathfrak{g} is rigid if and only if any perturbation \mathfrak{g}' (where the structure constants are in the ring of limited elements of the Robinson extension \mathbb{C}^* of \mathbb{C}) is (\mathbb{C}^*) -isomorphic to \mathfrak{g} .*

Example. Let us consider the Lie algebra $sl(2)$. It is the 3-dimensional Lie algebra given by

$$\begin{cases} \mu(e_1, e_2) = 2e_2 \\ \mu(e_1, e_3) = -2e_3 \\ \mu(e_2, e_3) = e_1. \end{cases}$$

This Lie algebra is simple. Let us prove that it is rigid. Let μ' be a perturbation of μ . Then the linear operator $ad_{\mu'}e_1$ (of the vector space $(\mathbb{C}^*)^3$) is a perturbation of $ad_{\mu}e_1$. Then the eigenvalue are close. This implies that the eigenvalue of $ad_{\mu'}e_1$ are $\epsilon_0, 2 + \epsilon_1, -2 + \epsilon_2$ where ϵ_i are in the maximal ideal, that is are infinitesimal. As $ad_{\mu'}e_1(e_1) = 0$, we have $\epsilon_0 = 0$. There exists a basis $\{e_1, e'_2, e'_3\}$ of $(\mathbb{C}^*)^3$ whose elements are eigenvectors of $ad_{\mu'}e_1$. We can always choose e'_2 such that $e'_2 - e_2$ has coefficients in the maximal ideal. Likewise for e'_3 . Thus $\mu'(e'_2, e'_3)$ is an eigenvector associated to $2 + \epsilon_1 - 2 + \epsilon_2 = \epsilon_1 + \epsilon_2$. But this vector is linearly dependant with e_1 . Then $\epsilon_1 + \epsilon_2 = 0$. We have then

$$\begin{cases} \mu'(e_1, e'_2) = 2(1 + \epsilon_1)e'_2 \\ \mu'(e_1, e'_3) = -2(1 + \epsilon_1)e'_3 \\ \mu'(e'_2, e'_3) = e_1. \end{cases}$$

If we put $e_1 = (1 + \epsilon_1)e'_1$ we obtain

$$\begin{cases} \mu'(e_1, e'_2) = 2e'_2 \\ \mu'(e_1, e'_3) = -2(1 + \epsilon'_3)e'_3 \\ \mu'(e'_2, e'_3) = e_1. \end{cases}$$

and μ' is isomorphic to μ . The Lie algebra $sl(2)$ is rigid.

We similar arguments, we can generalize this result :

Theorem 10 *Every complex simple or semi-simple Lie algebra is rigid.*

The proof can be read in [18].

8.3 Structure of rigid Lie algebras

Definition 21 *A complex (or real) Lie algebra \mathfrak{g} is called algebraic if \mathfrak{g} is isomorphic to a Lie algebra of a linear algebraic Lie group.*

A linear algebraic Lie group is a Lie subgroup of $GL(p, \mathbb{C})$ which is defined as the set of zeros of a finite system of polynomial equations. Contrary to the variety L_n , none of the points defined by this system is singular. This implies that an algebraic Lie group is a differential manifold.

Examples.

1. Every simple Lie algebra is algebraic.
2. Every nilpotent Lie algebra, or every Lie algebra whose radical is nilpotent is algebraic.

3. Every n -dimensional Lie algebra whose the Lie algebra of Derivations is also of dimension n is algebraic.

4. Every Lie algebra satisfying $\mathcal{D}^1(\mathfrak{g}) = \mathfrak{g}$ is algebraic.

We can see that the family of algebraic Lie algebras, for a given dimension, is very big and we have not general classification.

Proposition 21 *The following properties are equivalent :*

1. \mathfrak{g} is algebraic.
2. $ad(\mathfrak{g}) = \{ad_\mu X, X \in \mathfrak{g}\}$ is algebraic
3. $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n} \oplus \mathfrak{t}$ where \mathfrak{s} is the Levi semi-simple part, \mathfrak{n} the nilradical, that is the maximal nilpotent ideal and \mathfrak{t} a Malcev torus such that $ad_\mu \mathfrak{t}$ is algebraic.

In this proposition we have introduced the notion of Malcev torus. It is an abelian subalgebra \mathfrak{t} of \mathfrak{g} such that the endomorphisms adX , $X \in \mathfrak{t}$ are semi-simple (simultaneously diagonalizable). All the maximal torus for the inclusion are conjugated and their common dimension is called the rank.

Theorem 11 *Every rigid complex Lie algebra is algebraic.*

This theorem, proposed by R. Carles in [7] has many important consequence. For example, we prove in [2] that the the torus \mathfrak{t} is maximal. This permits to construct many rigid Lie algebra and to present some classification. For example in [12] we give the classification of rigid solvable Lie algebras whose nilradical is filiform. We give also the general classification of 8-dimensional rigid Lie algebras.

8.4 Rigidity and cohomology

This approach is more geometrical. If \mathfrak{g} is rigid, then its orbit is open. This implies that the tangent spaces to the orbit at μ and to ths schema \mathcal{L}_n always at μ coincide. Before we have determinated these vector spaces. We saw that they coincided with spaces $B^2(\mathfrak{g}, \mathfrak{g})$ and $Z^2(\mathfrak{g}, \mathfrak{g})$. We deduce the following Nijenhuis-Richardson theorem:

Theorem 12 *If $H^2(\mathfrak{g}, \mathfrak{g}) = 0$ then \mathfrak{g} is rigid.*

For example any semi simple Lie algebra has a trivial space of cohomology. Then it is rigid. Every rigid of dimension less or equal to 8 satisfies $H^2(\mathfrak{g}, \mathfrak{g}) = 0$. But there exists many examples of rigid Lie algebras which satisfy $H^2(\mathfrak{g}, \mathfrak{g}) \neq 0$. (see for example [12]). A geometric characterization of this fact is given in the following:

Proposition 22 *Let \mathfrak{g} be a rigid complex Lie algebra such that $H^2(\mathfrak{g}, \mathfrak{g}) \neq \{0\}$. Then the schema \mathcal{L}_n is not reduced at the point μ corresponding to \mathfrak{g} .*

Remark. This last time a study of real rigid Lie algebra has been proposed in [4].

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